TRANSMISSION AND REFLECTION OF ELECTROMAGNETIC WAVES IN RANDOMLY LAYERED MEDIA

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Abstract. In this paper the reflection of an obliquely incident electromagnetic wave on a randomly layered multiscale half-space is analyzed. By using homogenization and diffusion approximation theorems it is possible to get a complete description of the reflectivity of the random half-space that depends on the effective reflectivity of the interface and on the random reflectivity of the bulk medium. Particular attention is devoted to the characterization of the Brewster anomalies that correspond to small or even zero reflectivity. It turns out that the interface reflectivity and the random medium reflectivity as functions of the incidence angle may both possess Brewster angles that minimize or even cancel them, but these two angles are in general different.

Key words. Electromagnetic waves, random media, homogenization, diffusion approximation.
Subject classifications. 35Q61, 35R60, 78A48, 60F05.

This paper is dedicated to George Papanicolaou for his 70th birthday.

1. Introduction

Wave propagation in randomly layered media has been the subject of many papers in the last forty years, especially in the regime of separation of scales as introduced by George Papanicolaou and his coauthors [2, 4, 5, 15, 11]. It turns out that localization appears as a universal phenomenon, which implies in particular that a wave cannot propagate through a randomly layered medium beyond a critical distance called localization length and it is therefore completely reflected. However in some special situations it may happen that the localization length diverges. This phenomenon occurs for vector waves such as electromagnetic waves and it was first predicted in [17] and then confirmed by several other papers [1, 10, 13, 14]. In our paper we reproduce the original result and extend it by considering multiscale random media and by using the more recent asymptotic techniques presented for instance in [6]. In particular we deduce that multiscale random media, with fluctuations at the scale of the wavelength and at smaller scales, may produce a more complex picture, and that there may be a non-trivial interplay between the interface reflectivity (between the homogeneous half-space and the randomly layered one) and the bulk medium reflectivity. It turns out that both reflectivities have Brewster anomalies, corresponding to vanishing reflectivities, but not at the same angle in general. As a result, it is usually not possible to achieve zero reflection by a randomly layered medium. These results could find applications to wave propagation through foliage [7, 8] in which Brewster anomalies are sought for deeper penetration.

The paper is organized as follows. In Section 2 we introduce the model and identify the scales present in the problem. Then we transform the governing equations and identify the mode decomposition that is useful for the analysis in Section 3. We study the reflection and transmission problem for small propagation times and distances in Section 4. For this case we then identify Brewster anomalies for the effective (homogenized) interface problem. We consider the situation with large propagation

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times and distances in Sections 5-6, where we then can discuss Brewster anomalies for the localization problem.

2. Maxwell's equations for electromagnetic waves in a multiscale random medium We consider electromagnetic waves propagating in a three-dimensional medium with heterogeneous one-dimensional random fluctuations. Maxwell's equations for the electric field $\boldsymbol{E}(t,\boldsymbol{x},z)$ and the magnetic field $\boldsymbol{H}(t,\boldsymbol{x},z)$ are

$$\nabla \times \boldsymbol{E} = -\mu_0 \partial_t \boldsymbol{H}, \tag{2.1}$$
$$\nabla \cdot (\varepsilon(z)\boldsymbol{E}) = 0, \tag{2.2}$$
$$\nabla \times \boldsymbol{H} = \sigma(z)\boldsymbol{E} + \varepsilon(z)\partial_t \boldsymbol{E}, \tag{2.3}$$
$$\nabla \cdot (\mu_0 \boldsymbol{H}) = 0. \tag{2.4}$$

Here we denote the spatial variable by $(\boldsymbol{x},z)$, with $\boldsymbol{x}=(x,y)\in\mathbb{R}^2$.

We consider the high-frequency regime in which the propagation distance is larger than the wavelength. The medium has two scales of variations: rapid and strong fluctuations (with a correlation length smaller than the typical wavelength) and slow and weak fluctuations (with a correlation length of the same order as the typical wavelength). We also take into account small dissipative terms that can be homogeneous or heterogeneous.

We consider in this paper the situation in which a wave coming from the homogeneous half-space $z<0$ is impinging on an heterogeneous half-space $z>0$. We denote the medium permittivity in the half-space $z<0$ by ε_0. The medium parameters are:

$$\varepsilon(z) = \begin{cases} \varepsilon_0 & \text{if } z<0, \\ \varepsilon_{\rm r}\big(\frac{z}{\delta}\big) + \sqrt{\delta}\varepsilon_{\rm w}(z) & \text{if } z>0, \end{cases} \tag{2.5}$$

$$\sigma(z) = \begin{cases} 0 & \text{if } z<0, \\ \delta\varepsilon_{\rm d}\big(\frac{z}{\delta}\big) & \text{if } z>0, \end{cases} \tag{2.6}$$

where δ is a small dimensionless parameter that characterizes the scaling ratios between the different types of fluctuations and the typical wavelength which is our reference length scale of order one. The random processes $\varepsilon_{\rm r}(z)$ and $\varepsilon_{\rm w}(z)$ model the fluctuations of the permittivity of the medium. $\varepsilon_{\rm r}(z)$ models rapid and strong fluctuations, $\varepsilon_{\rm w}(z)$ models slow and weak fluctuations. The random process $\varepsilon_{\rm d}(z)$ models the dissipation, which has small amplitude and may be constant or spatially fluctuating. We assume that these processes are bounded, stationary, and that they satisfy strong mixing conditions. For simplicity we also assume that they are independent.

We shall refer to waves propagating in a direction with a negative (resp. positive) z-component as left-going (resp. right-going) waves.

For illustration we will consider two particular models for the random process $\varepsilon_{\rm r}$.

Model I (binary medium). The first model is a binary medium made of two materials with permittivity ε_0 and ε_1. The lengths of the elementary intervals (over which the permittivity is constant) are independent. The lengths of the intervals of material 0 are distributed with the exponential distribution with mean $l_{\rm c}/\alpha$. The lengths of the intervals of material 1 are distributed with the exponential distribution with mean $l_{\rm c}/(1-\alpha)$. Here $l_{\rm c}>0$ and $\alpha\in(0,1)$. At $z=0$ the medium is made of material 0 with probability $1-\alpha$ and of material 1 with probability α (see Figure 2.1a for a realization).

The process $(\varepsilon_{\rm r}(z))_{z\geq 0}$ is a Markov process with state space $\{\varepsilon_0,\varepsilon_1\}$ and with infinitesimal generator:

$$\mathcal{L}_{\rm I} f(\varepsilon)=\Big[-\frac{\alpha}{l_{\rm c}}f(\varepsilon_0)+\frac{\alpha}{l_{\rm c}}f(\varepsilon_1)\Big]\mathbf{1}_{\varepsilon_0}(\varepsilon)+\Big[\frac{1-\alpha}{l_{\rm c}}f(\varepsilon_0)-\frac{1-\alpha}{l_{\rm c}}f(\varepsilon_1)\Big]\mathbf{1}_{\varepsilon_1}(\varepsilon),\quad \varepsilon\in\{\varepsilon_0,\varepsilon_1\},$$

whose adjoint is

$$\mathcal{L}_{\rm I}^* p(\varepsilon)=\Big[-\frac{\alpha}{l_{\rm c}}p(\varepsilon_0)+\frac{1-\alpha}{l_{\rm c}}p(\varepsilon_1)\Big]\mathbf{1}_{\varepsilon_0}(\varepsilon)+\Big[\frac{\alpha}{l_{\rm c}}p(\varepsilon_0)-\frac{1-\alpha}{l_{\rm c}}p(\varepsilon_1)\Big]\mathbf{1}_{\varepsilon_1}(\varepsilon).$$

The unique invariant probability measure (that satisfies $\mathcal{L}_{\rm I}^* p=0$) is $p(\varepsilon_0)=1-\alpha$ and $p(\varepsilon_1)=\alpha$. Starting from this probability measure, the process is stationary, and the volume fraction of material 1 is α. The expectation and the variance are therefore

$$\mathbb{E}[\varepsilon_{\rm r}(0)]=\alpha\varepsilon_1+(1-\alpha)\varepsilon_0,\qquad {\rm Var}(\varepsilon_{\rm r}(0))=\alpha(1-\alpha)(\varepsilon_1-\varepsilon_0)^2.$$

Furthermore, denoting $g(\varepsilon)=\varepsilon-\mathbb{E}[\varepsilon_{\rm r}(0)]$, we have $\mathcal{L}_{\rm I} g(\varepsilon)=-g(\varepsilon)/l_{\rm c}$. Using the fact that $g(\varepsilon_{\rm r}(z))-\int_0^z \mathcal{L}_{\rm I} g(\varepsilon_{\rm r}(z'))dz'$ is a martingale and that ${\rm Cov}(\varepsilon_{\rm r}(0),\varepsilon_{\rm r}(z))=\mathbb{E}[g(\varepsilon_{\rm r}(z))\varepsilon_{\rm r}(0))]$, we get

$$\partial_z {\rm Cov}(\varepsilon_{\rm r}(0),\varepsilon_{\rm r}(z))=-\frac{1}{l_{\rm c}}{\rm Cov}(\varepsilon_{\rm r}(0),\varepsilon_{\rm r}(z)),$$

which gives

$${\rm Cov}(\varepsilon_{\rm r}(0),\varepsilon_{\rm r}(z))=\exp(-|z|/l_{\rm c}){\rm Var}(\varepsilon_{\rm r}(0)),$$

which shows that the correlation length of the medium is $l_{\rm c}$.

Model II (uniform medium). The second model is a stepwise constant medium. The lengths of the elementary intervals are independent and uniformly distributed with the exponential distribution with mean $l_{\rm c}$. The values of the permittivities in the intervals are independent and uniformly distributed with the uniform distribution over $(\varepsilon_0,\varepsilon_1)$ (see Figure 2.1b for a realization).
The process $(\varepsilon_{\rm r}(z))_{z\geq 0}$ is a Markov process with state space $[\varepsilon_0,\varepsilon_1]$ and with infinitesimal generator:

$$\mathcal{L}_{\rm II} f(\varepsilon)=\frac{1}{l_c}\Big[-f(\varepsilon)+\frac{1}{\varepsilon_1-\varepsilon_0}\int_{\varepsilon_0}^{\varepsilon_1} f(\varepsilon')d\varepsilon'\Big],\quad \varepsilon\in[\varepsilon_0,\varepsilon_1],$$

which is self-adjoint. The unique invariant probability measure (that satisfies $\mathcal{L}_{\rm II}^* p=0$), is the uniform measure over $[\varepsilon_0,\varepsilon_1]$. Starting from this probability measure, the process is stationary, and the volume fraction of material 1 is 1/2. The expectation and the variance are therefore

$$\mathbb{E}[\varepsilon_{\rm r}(0)]=(\varepsilon_1+\varepsilon_0)/2,\qquad {\rm Var}(\varepsilon_{\rm r}(0))=(\varepsilon_1-\varepsilon_0)^2/12.$$

Furthermore, denoting $g(\varepsilon)=\varepsilon-\mathbb{E}[\varepsilon_{\rm r}(0)]$, we have $\mathcal{L}_{\rm II} g(\varepsilon)=-g(\varepsilon)/l_{\rm c}$. We get as in model I that

$${\rm Cov}(\varepsilon_{\rm r}(0),\varepsilon_{\rm r}(z))=\exp(-|z|/l_{\rm c}){\rm Var}(\varepsilon_{\rm r}(0)),$$

which shows that the correlation length of the medium is $l_{\rm c}$.

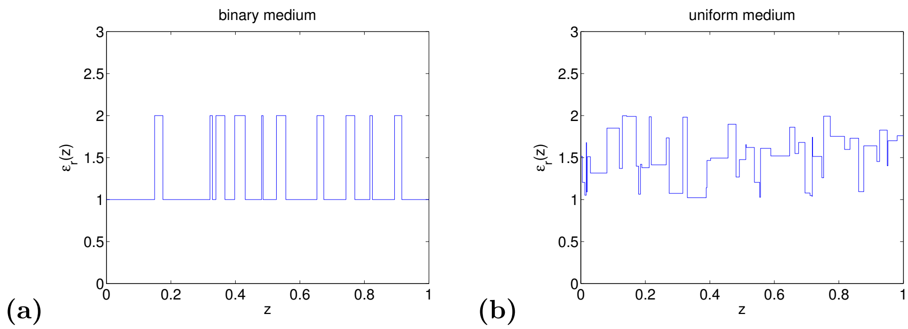


FIG. 2.1. *Realizations of the permittivity of the random medium for the two models. Here* $l_c = 0.02$, $\varepsilon_0 = 1$, $\varepsilon_1 = 2$, $\alpha = 0.25$ *(model I, binary medium, left picture) and* $l_c = 0.02$, $\varepsilon_0 = 1$, $\varepsilon_1 = 2$ *(model II, uniform medium, right picture).*

3. The reduced wave equations for the wave modes We take a Fourier transform in time and lateral spatial coordinates:

$$\hat{\boldsymbol{E}}(\omega,\boldsymbol{\kappa},z) = \int_{\mathbb{R}^2}\int_{\mathbb{R}} \boldsymbol{E}(t,\boldsymbol{x},z) e^{i\omega t - i\omega\boldsymbol{\kappa}\cdot\boldsymbol{x}} \mathrm{d}t \mathrm{d}\boldsymbol{x}, \tag{3.1}$$

$$\hat{\boldsymbol{H}}(\omega,\boldsymbol{\kappa},z) = \int_{\mathbb{R}^2}\int_{\mathbb{R}} \boldsymbol{H}(t,\boldsymbol{x},z) e^{i\omega t - i\omega\boldsymbol{\kappa}\cdot\boldsymbol{x}} \mathrm{d}t \mathrm{d}\boldsymbol{x}. \tag{3.2}$$

We denote $\hat{\boldsymbol{E}} = (\hat{E}_j)_{j=1,2,3}$ and $\hat{\boldsymbol{H}} = (\hat{H}_j)_{j=1,2,3}$. The four-dimensional vector $(\hat{E}_1, \hat{H}_2, \hat{E}_2, \hat{H}_1)$ satisfies:

$$\partial_z \hat{E}_1 = i\omega\mu_0 \hat{H}_2 - \frac{i\omega}{\varepsilon(z,\omega)}\kappa_1\big(\kappa_1\hat{H}_2 - \kappa_2\hat{H}_1\big), \tag{3.3}$$

$$\partial_z \hat{H}_2 = i\omega\varepsilon(z,\omega)\hat{E}_1 + \frac{i\omega}{\mu_0}\kappa_2\big(\kappa_1\hat{E}_2 - \kappa_2\hat{E}_1\big), \tag{3.4}$$

$$\partial_z \hat{E}_2 = -i\omega\mu_0 \hat{H}_1 - \frac{i\omega}{\varepsilon(z,\omega)}\kappa_2\big(\kappa_1\hat{H}_2 - \kappa_2\hat{H}_1\big), \tag{3.5}$$

$$\partial_z \hat{H}_1 = -i\omega\varepsilon(z,\omega)\hat{E}_2 + \frac{i\omega}{\mu_0}\kappa_1\big(\kappa_1\hat{E}_2 - \kappa_2\hat{E}_1\big), \tag{3.6}$$

where

$$\varepsilon(z,\omega) = \begin{cases} \varepsilon_0 & \text{if } z<0, \\ \varepsilon_{\mathrm{r}}\big(\frac{z}{\delta}\big) + \sqrt{\delta}\varepsilon_{\mathrm{w}}(z) + \delta\frac{i}{\omega}\varepsilon_{\mathrm{d}}\big(\frac{z}{\delta}\big) & \text{if } z>0. \end{cases} \tag{3.7}$$

The two other variables $(\hat{E}_3, \hat{H}_3)$ are given by:

$$\hat{E}_3 = -\frac{1}{\varepsilon(z,\omega)}(\kappa_1\hat{H}_2 - \kappa_2\hat{H}_1), \tag{3.8}$$

$$\hat{H}_3 = \frac{1}{\mu_0}(\kappa_1\hat{E}_2 - \kappa_2\hat{E}_1). \tag{3.9}$$

Let us make the change of variables:

$$\hat{H}_{\rm p} = \frac{\kappa_1 \hat{H}_2 - \kappa_2 \hat{H}_1}{|\boldsymbol{\kappa}|}, \tag{3.10}$$

$$\hat{E}_{\rm p} = \frac{\kappa_1 \hat{E}_1 + \kappa_2 \hat{E}_2}{|\boldsymbol{\kappa}|}, \tag{3.11}$$

$$\hat{H}_{\rm s} = \frac{\kappa_1 \hat{H}_1 + \kappa_2 \hat{H}_2}{|\boldsymbol{\kappa}|}, \tag{3.12}$$

$$\hat{E}_{\rm s} = \frac{\kappa_1 \hat{E}_2 - \kappa_2 \hat{E}_1}{|\boldsymbol{\kappa}|}. \tag{3.13}$$

If $\boldsymbol{\kappa} = \mathbf{0}$, we use the convention $\hat{H}_{\rm p} = \hat{H}_2$, $\hat{E}_{\rm p} = \hat{E}_1$, $\hat{H}_{\rm s} = \hat{H}_1$, $\hat{E}_{\rm s} = \hat{E}_2$. This coordinate system is frequently used. It relates to the plane of incidence, that is, the plane made by the propagation axis and the normal vector to the surface $z=0$. Here it is the plane that contains the vectors $(\mathbf{0},1)$ and $(\boldsymbol{\kappa},0)$. $\hat{E}_{\rm s}$, resp. $\hat{H}_{\rm p}$, is the component of the electric field, resp. magnetic field, that is perpendicular to the incidence plane. The pairs $(\hat{E}_{\rm p},\hat{H}_{\rm p})$ and $(\hat{E}_{\rm s},\hat{H}_{\rm s})$ satisfy the uncoupled systems of equations:

$$\left\{ \begin{array}{l} \partial_z \hat{H}_{\rm p} = i\omega\varepsilon(z,\omega)\hat{E}_{\rm p}, \\ \partial_z \hat{E}_{\rm p} = i\omega\mu_0 \Big(1 - \dfrac{|\boldsymbol{\kappa}|^2}{\mu_0\varepsilon(z,\omega)}\Big)\hat{H}_{\rm p}, \end{array} \right. \tag{3.14}$$

$$\left\{ \begin{array}{l} \partial_z \hat{H}_{\rm s} = -i\omega\varepsilon(z,\omega)\Big(1 - \dfrac{|\boldsymbol{\kappa}|^2}{\mu_0\varepsilon(z,\omega)}\Big)\hat{E}_{\rm s}, \\ \partial_z \hat{E}_{\rm s} = -i\omega\mu_0 \hat{H}_{\rm s}. \end{array} \right. \tag{3.15}$$

The pair $(\hat{E}_{\rm p},\hat{H}_{\rm p})$, resp. $(\hat{E}_{\rm s},\hat{H}_{\rm s})$, represents the p-polarized, resp. s-polarized, component of the electromagnetic wave. If a wave is purely p-polarized, i.e. $\hat{E}_{\rm s} = \hat{H}_{\rm s} = 0$, then we also have $\hat{H}_3 = 0$ by (3.9), which means that the projection of the magnetic field on the incidence plane is zero, and the wave is transverse magnetic. If a wave is purely s-polarized, i.e. $\hat{E}_{\rm p} = \hat{H}_{\rm p} = 0$, then we also have $\hat{E}_3 = 0$ by (3.8), which means that the projection of the electric field on the incidence plane is zero, and the wave is transverse electric.

If the medium is homogeneous with permittivity $\varepsilon(z,\omega) \equiv \varepsilon_0$, then each of the field components $\{\hat{E}_{\rm p},\hat{H}_{\rm p},\hat{E}_{\rm s},\hat{H}_{\rm s}\}$ solves the scalar wave equation

$$\partial_z^2 \hat{U} + \omega^2 \big(c_0^{-2} - |\boldsymbol{\kappa}|^2\big)\hat{U} = 0, \tag{3.16}$$

where $c_0 = \mu_0^{-1/2}\varepsilon_0^{-1/2}$ is the homogeneous wave speed and the slowness in the homogeneous medium is

$$\kappa_0(\boldsymbol{\kappa}) = \sqrt{c_0^{-2} - |\boldsymbol{\kappa}|^2}. \tag{3.17}$$

Consequently, each of the field components $\{\hat{E}_{\rm p},\hat{H}_{\rm p},\hat{E}_{\rm s},\hat{H}_{\rm s}\}$ is a linear combination of the two linearly independent solutions $e^{i\omega\kappa_0(\boldsymbol{\kappa})z}$ and $e^{-i\omega\kappa_0(\boldsymbol{\kappa})z}$ of (3.16), which are respectively a rigth-going wave and a left-going wave. The general solution of the system (3.14-3.15) in the homogeneous case $\varepsilon(z,\omega) \equiv \varepsilon_0$ can therefore be written in the form

$$\hat{H}_{\rm p}(\omega,\boldsymbol{\kappa},z)=\hat{a}_{\rm p}(\omega,\boldsymbol{\kappa})e^{i\omega\kappa_0(\boldsymbol{\kappa})z}+\hat{b}_{\rm p}(\omega,\boldsymbol{\kappa})e^{-i\omega\kappa_0(\boldsymbol{\kappa})z}, \tag{3.18}$$

$$\hat{E}_{\rm p}(\omega,\boldsymbol{\kappa},z)=\frac{\kappa_0(\boldsymbol{\kappa})}{\varepsilon_0}\Big(\hat{a}_{\rm p}(\omega,\boldsymbol{\kappa})e^{i\omega\kappa_0(\boldsymbol{\kappa})z}-\hat{b}_{\rm p}(\omega,\boldsymbol{\kappa})e^{-i\omega\kappa_0(\boldsymbol{\kappa})z}\Big), \tag{3.19}$$

$$\hat{H}_{\rm s}(\omega,\boldsymbol{\kappa},z)=\hat{a}_{\rm s}(\omega,\boldsymbol{\kappa})e^{i\omega\kappa_0(\boldsymbol{\kappa})z}+\hat{b}_{\rm s}(\omega,\boldsymbol{\kappa})e^{-i\omega\kappa_0(\boldsymbol{\kappa})z}, \tag{3.20}$$

$$\hat{E}_{\rm s}(\omega,\boldsymbol{\kappa},z)=\frac{\mu_0}{\kappa_0(\boldsymbol{\kappa})}\Big(-\hat{a}_{\rm s}(\omega,\boldsymbol{\kappa})e^{i\omega\kappa_0(\boldsymbol{\kappa})z}+\hat{b}_{\rm s}(\omega,\boldsymbol{\kappa})e^{-i\omega\kappa_0(\boldsymbol{\kappa})z}\Big). \tag{3.21}$$

Here $\hat{a}_{\rm p}$, $\hat{a}_{\rm s}$ are the complex amplitudes of the right-going p- and s-waves, while $\hat{b}_{\rm p}$, $\hat{b}_{\rm s}$ are the complex amplitudes of the left-going p- and s-waves.

If the medium is as described by (2.5), then the wave is of the form (3.18-3.21) in the region $z<0$ and it satisfies (3.14-3.15) in the region $z>0$. The system is complemented by jump conditions at $z=0$ that express the continuity of the transverse electromagnetic field. If there is an incoming, right-going wave that we denote $(\hat{\boldsymbol{E}}_{\rm inc},\hat{\boldsymbol{H}}_{\rm inc})$ and that is coming from the left half-space $z<0$, then the boundary conditions at $z=0$ read

$$\hat{H}_{\rm p}(\omega,\boldsymbol{\kappa},0)+\frac{\varepsilon_0}{\kappa_0(\boldsymbol{\kappa})}\hat{E}_{\rm p}(\omega,\boldsymbol{\kappa},0)=2\hat{a}_{\rm inc,p}(\omega,\boldsymbol{\kappa}), \tag{3.22}$$

$$\hat{H}_{\rm s}(\omega,\boldsymbol{\kappa},0)-\frac{\kappa_0(\boldsymbol{\kappa})}{\mu_0}\hat{E}_{\rm s}(\omega,\boldsymbol{\kappa},0)=2\hat{a}_{\rm inc,s}(\omega,\boldsymbol{\kappa}). \tag{3.23}$$

This comes from the continuity accross the interface $z=0$ of the transverse electromagnetic field $(\hat{E}_1,\hat{E}_2,\hat{H}_1,\hat{H}_2)$, or equivalently $(\hat{E}_{\rm p},\hat{E}_{\rm s},\hat{H}_{\rm p},\hat{H}_{\rm s})$, and from the form of the right-propagating incoming wave in the half-space $z<0$:

$$\hat{H}_{\rm inc,p}(\omega,\boldsymbol{\kappa},z)=\hat{a}_{\rm inc,p}(\omega,\boldsymbol{\kappa})e^{i\omega\kappa_0(\boldsymbol{\kappa})z}, \tag{3.24}$$

$$\hat{E}_{\rm inc,p}(\omega,\boldsymbol{\kappa},z)=\frac{\kappa_0(\boldsymbol{\kappa})}{\varepsilon_0}\hat{a}_{\rm inc,p}(\omega,\boldsymbol{\kappa})e^{i\omega\kappa_0(\boldsymbol{\kappa})z}, \tag{3.25}$$

$$\hat{H}_{\rm inc,s}(\omega,\boldsymbol{\kappa},z)=\hat{a}_{\rm inc,s}(\omega,\boldsymbol{\kappa})e^{i\omega\kappa_0(\boldsymbol{\kappa})z}, \tag{3.26}$$

$$\hat{E}_{\rm inc,s}(\omega,\boldsymbol{\kappa},z)=-\frac{\mu_0}{\kappa_0(\boldsymbol{\kappa})}\hat{a}_{\rm inc,s}(\omega,\boldsymbol{\kappa})e^{i\omega\kappa_0(\boldsymbol{\kappa})z}. \tag{3.27}$$

The incident p- and s-wave mode amplitudes $\hat{a}_{\rm inc,p}$ and $\hat{a}_{\rm inc,s}$ can be expressed as

$$\begin{aligned}\hat{a}_{\rm inc,p}(\omega,\boldsymbol{\kappa})&=\frac{(\kappa_1\hat{H}_{\rm inc,2}-\kappa_2\hat{H}_{\rm inc,1})(\omega,\boldsymbol{\kappa},0)}{|\boldsymbol{\kappa}|}\\&=\frac{\varepsilon_0}{\kappa_0(\boldsymbol{\kappa})}\frac{(\kappa_1\hat{E}_{\rm inc,1}+\kappa_2\hat{E}_{\rm inc,2})(\omega,\boldsymbol{\kappa},0)}{|\boldsymbol{\kappa}|},\end{aligned} \tag{3.28}$$

$$\begin{aligned}\hat{a}_{\rm inc,s}(\omega,\boldsymbol{\kappa})&=\frac{(\kappa_1\hat{H}_{\rm inc,1}+\kappa_2\hat{H}_{\rm inc,2})(\omega,\boldsymbol{\kappa},0)}{|\boldsymbol{\kappa}|}\\&=-\frac{\kappa_0(\boldsymbol{\kappa})}{\mu_0}\frac{(\kappa_1\hat{E}_{\rm inc,2}-\kappa_2\hat{E}_{\rm inc,1})(\omega,\boldsymbol{\kappa},0)}{|\boldsymbol{\kappa}|}.\end{aligned} \tag{3.29}$$

We should also impose radiation conditions at $z\to\infty$. For this, a convenient way is to choose a depth L^δ that is large enough so that an incoming wave from $z<0$ does not reach that depth on the duration of the experiment. This is always possible by hyperbolicity of the wave equation. We can then impose an arbitrary boundary condition at $z=L^\delta$. It is convenient for the forthcoming analysis to choose to impose radiation conditions of the form

$$\hat{H}_{\rm p}(\omega,\boldsymbol{\kappa},L^\delta) - \frac{\varepsilon_{\rm a}}{\kappa_{\rm p}(\boldsymbol{\kappa})}\hat{E}_{\rm p}(\omega,\boldsymbol{\kappa},L^\delta) = 0, \tag{3.30}$$

$$\hat{H}_{\rm s}(\omega,\boldsymbol{\kappa},L^\delta) + \frac{\kappa_{\rm s}(\boldsymbol{\kappa})}{\mu_0}\hat{E}_{\rm s}(\omega,\boldsymbol{\kappa},L^\delta) = 0, \tag{3.31}$$

where the wavenumbers $\kappa_{\rm s}(\boldsymbol{\kappa})$, $\kappa_{\rm p}(\boldsymbol{\kappa})$, and the effective permittivity $\varepsilon_{\rm a}$ are defined in the next section. It turns out that these conditions are transparent at the interface $z=L^\delta$ in the regime $\delta\to 0$, see (5.5)-(5.8). We remark that the "transparent" terminating condition that we have introduced can be interpreted physically as introducing an "annihilation" source in the (assumed homogeneous) half space $z>L$ that cancels the wave reflections from the interface at $z=L$. As we will see below we can not synthesize such transparent boundary conditions simply by choosing the values of the homogeneous parameters in $z>L$ since the homogenized parameters in the section $[0,L]$ depends on both arithmetic and harmonic means of the fluctuating permittivity.

It can be seen from the system (3.14-3.15) that the p-wave $(\hat{E}_{\rm p},\hat{H}_{\rm p})$ and s-wave $(\hat{E}_{\rm s},\hat{H}_{\rm s})$ propagate without interaction in the heterogeneous medium. However these waves interact with the same realization of the random medium, so that their dynamics are in fact coupled from the statistical point of view. Indeed we will see in the following that, amongst other phenomena, the p- and s-waves experience random travel time corrections which are correlated.

4. Transmission and reflection in the homogenization regime Let us assume that the propagation time is short, of the order of the pulse width. We can then take $L^\delta=L$, that is to say, we consider a heterogeneous region whose size is of the same order as the wavelength. The homogenization theory applies [16],[6, Chapter 4] and we find the effective wave equations inside the inhomogeneous region

$$\begin{cases} \partial_z \hat{H}_{\rm p} = i\omega\varepsilon_{\rm a}\hat{E}_{\rm p}, \\ \partial_z \hat{E}_{\rm p} = i\omega\mu_0\Big(1-\dfrac{|\boldsymbol{\kappa}|^2}{\mu_0\varepsilon_{\rm h}}\Big)\hat{H}_{\rm p}, \end{cases} \tag{4.1}$$

$$\begin{cases} \partial_z \hat{H}_{\rm s} = -i\omega\varepsilon_{\rm a}\Big(1-\dfrac{|\boldsymbol{\kappa}|^2}{\mu_0\varepsilon_{\rm a}}\Big)\hat{E}_{\rm s}, \\ \partial_z \hat{E}_{\rm s} = -i\omega\mu_0\hat{H}_{\rm s}, \end{cases} \tag{4.2}$$

where $\varepsilon_{\rm a}$ and $\varepsilon_{\rm h}$ are the arithmetic and harmonic averages of the stationary process $\varepsilon_{\rm r}(z)$:

$$\varepsilon_{\rm a} = \mathbb{E}[\varepsilon_{\rm r}(0)], \qquad \varepsilon_{\rm h} = \mathbb{E}[\varepsilon_{\rm r}(0)^{-1}]^{-1}. \tag{4.3}$$

For model I (for which the stationary measure is $p(\varepsilon_0)=1-\alpha$ and $p(\varepsilon_1)=\alpha$), we have

$$\varepsilon_{\rm a} = \alpha\varepsilon_1 + (1-\alpha)\varepsilon_0, \qquad \varepsilon_{\rm h} = \frac{\varepsilon_0\varepsilon_1}{\alpha\varepsilon_0+(1-\alpha)\varepsilon_1}.$$

For model II (for which the stationary measure is the uniform measure over $[\varepsilon_0,\varepsilon_1]$), we have

$$\varepsilon_{\rm a} = \frac{\varepsilon_1+\varepsilon_0}{2}, \qquad \varepsilon_{\rm h} = \frac{\varepsilon_1-\varepsilon_0}{\log(\varepsilon_1/\varepsilon_0)}.$$

Note that we always have $\varepsilon_{\rm a} \geq \varepsilon_{\rm h}$ (by Cauchy-Schwarz inequality).

The wavenumbers of a time-harmonic plane p-wave or s-wave with frequency ω and slowness vector $\boldsymbol{\kappa}$ are $k_{\rm p} = \omega\kappa_{\rm p}(\boldsymbol{\kappa})$ and $k_{\rm s} = \omega\kappa_{\rm s}(\boldsymbol{\kappa})$ with

$$\kappa_{\rm p}^2(\boldsymbol{\kappa}) = \mu_0\varepsilon_{\rm a} - |\boldsymbol{\kappa}|^2\frac{\varepsilon_{\rm a}}{\varepsilon_{\rm h}}, \qquad \kappa_{\rm s}^2(\boldsymbol{\kappa}) = \mu_0\varepsilon_{\rm a} - |\boldsymbol{\kappa}|^2. \tag{4.4}$$

Remember that the p- and s-waves in the homogeneous region with permittivity ε_0 have the same wavenumber $\omega\kappa_0(\boldsymbol{\kappa})$.

The system (4.1-4.2) is complemented with the boundary conditions (3.22-3.23) at $z=0$ and (3.30-3.31) at $z=L$. Note that the boundary conditions at $z=L$ now appear as transparent since they impose that the left-going waves in the homogenized medium at $z=L$ are vanishing. In the case when an incoming plane wave arrives from the half-space $z<0$, it is is partly reflected and partly transmitted. The p-wave, respectively s-wave, component with frequency ω and slowness vector $\boldsymbol{\kappa}$ have the form

$$\hat{H}_{\rm p}(\omega,\boldsymbol{\kappa},z) = \left[e^{i\omega\kappa_0(\boldsymbol{\kappa})z} + R^{\rm i}_{\rm p}(\boldsymbol{\kappa})e^{-i\omega\kappa_0(\boldsymbol{\kappa})z}\right]\hat{H}_{\rm inc,p}(\omega,\boldsymbol{\kappa},0), \tag{4.5}$$

$$\hat{H}_{\rm s}(\omega,\boldsymbol{\kappa},z) = \left[e^{i\omega\kappa_0(\boldsymbol{\kappa})z} + R^{\rm i}_{\rm s}(\boldsymbol{\kappa})e^{-i\omega\kappa_0(\boldsymbol{\kappa})z}\right]\hat{H}_{\rm inc,s}(\omega,\boldsymbol{\kappa},0), \tag{4.6}$$

$$\hat{E}_{\rm p}(\omega,\boldsymbol{\kappa},z) = \left[e^{i\omega\kappa_0(\boldsymbol{\kappa})z} - R^{\rm i}_{\rm p}(\boldsymbol{\kappa})e^{-i\omega\kappa_0(\boldsymbol{\kappa})z}\right]\hat{E}_{\rm inc,p}(\omega,\boldsymbol{\kappa},0), \tag{4.7}$$

$$\hat{E}_{\rm s}(\omega,\boldsymbol{\kappa},z) = \left[e^{i\omega\kappa_0(\boldsymbol{\kappa})z} - R^{\rm i}_{\rm s}(\boldsymbol{\kappa})e^{-i\omega\kappa_0(\boldsymbol{\kappa})z}\right]\hat{E}_{\rm inc,s}(\omega,\boldsymbol{\kappa},0), \tag{4.8}$$

in the region $z<0$, and

$$\hat{H}_{\rm p}(\omega,\boldsymbol{\kappa},z) = T^{\rm i}_{\rm p}(\boldsymbol{\kappa})e^{i\omega\kappa_{\rm p}(\boldsymbol{\kappa})z}\hat{H}_{\rm inc,p}(\omega,\boldsymbol{\kappa},0), \tag{4.9}$$

$$\hat{H}_{\rm s}(\omega,\boldsymbol{\kappa},z) = T^{\rm i}_{\rm s}(\boldsymbol{\kappa})e^{i\omega\kappa_{\rm s}(\boldsymbol{\kappa})z}\hat{H}_{\rm inc,s}(\omega,\boldsymbol{\kappa},0), \tag{4.10}$$

$$\hat{E}_{\rm p}(\omega,\boldsymbol{\kappa},z) = \frac{\varepsilon_0\kappa_{\rm p}(\boldsymbol{\kappa})}{\varepsilon_{\rm a}\kappa_0(\boldsymbol{\kappa})}T^{\rm i}_{\rm p}(\boldsymbol{\kappa})e^{i\omega\kappa_{\rm p}(\boldsymbol{\kappa})z}\hat{E}_{\rm inc,p}(\omega,\boldsymbol{\kappa},0), \tag{4.11}$$

$$\hat{E}_{\rm s}(\omega,\boldsymbol{\kappa},z) = \frac{\kappa_0(\boldsymbol{\kappa})}{\kappa_{\rm s}(\boldsymbol{\kappa})}T^{\rm i}_{\rm s}(\boldsymbol{\kappa})e^{i\omega\kappa_{\rm s}(\boldsymbol{\kappa})z}\hat{E}_{\rm inc,s}(\omega,\boldsymbol{\kappa},0), \tag{4.12}$$

in the region $z>0$. The reflection and transmission coefficients for the interface $z=0$ (obtained from the continuity of the electromagnetic field) are given by

$$R^{\rm i}_{\rm p}(\boldsymbol{\kappa}) = \frac{\frac{\kappa_0(\boldsymbol{\kappa})}{\varepsilon_0} - \frac{\kappa_{\rm p}(\boldsymbol{\kappa})}{\varepsilon_{\rm a}}}{\frac{\kappa_0(\boldsymbol{\kappa})}{\varepsilon_0} + \frac{\kappa_{\rm p}(\boldsymbol{\kappa})}{\varepsilon_{\rm a}}}, \tag{4.13}$$

$$R^{\rm i}_{\rm s}(\boldsymbol{\kappa}) = \frac{\kappa_{\rm s}(\boldsymbol{\kappa}) - \kappa_0(\boldsymbol{\kappa})}{\kappa_{\rm s}(\boldsymbol{\kappa}) + \kappa_0(\boldsymbol{\kappa})}, \tag{4.14}$$

and with

$$T^{\rm i}_q(\boldsymbol{\kappa}) = 1 + R^{\rm i}_q(\boldsymbol{\kappa}), \quad q = {\rm p,s}. \tag{4.15}$$

Note also that we have the relations

$$\frac{\kappa_0(\boldsymbol{\kappa})}{\varepsilon_0}|R^{\rm i}_{\rm p}(\boldsymbol{\kappa})|^2 + \frac{\kappa_{\rm p}(\boldsymbol{\kappa})}{\varepsilon_{\rm a}}|T^{\rm i}_{\rm p}(\boldsymbol{\kappa})|^2 = \frac{\kappa_0(\boldsymbol{\kappa})}{\varepsilon_0}, \tag{4.16}$$

$$\frac{\mu_0}{\kappa_0(\boldsymbol{\kappa})}|R^{\rm i}_{\rm s}(\boldsymbol{\kappa})|^2 + \frac{\mu_0}{\kappa_{\rm s}(\boldsymbol{\kappa})}|T^{\rm i}_{\rm s}(\boldsymbol{\kappa})|^2 = \frac{\mu_0}{\kappa_0(\boldsymbol{\kappa})}. \tag{4.17}$$

These relations can be interpreted as energy conservation relations. Indeed the energy flux density (the rate of energy transfer per unit area) of an electromagnetic wave with

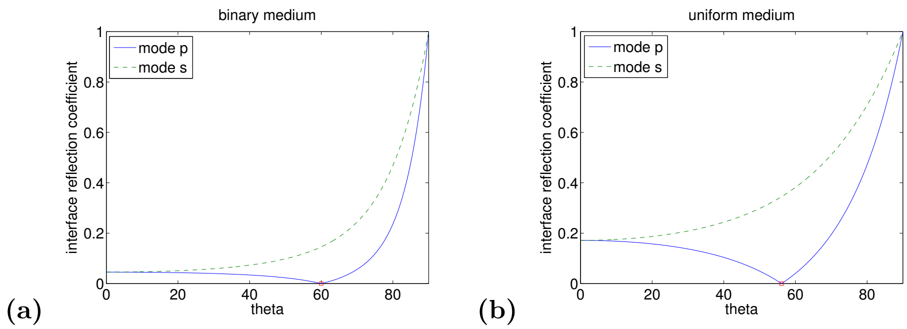


FIG. 4.1. *Reflection coefficient of the interface for the two models and the p,s-waves as a function of the incidence angle θ (with $|\boldsymbol{\kappa}|c_0 = \sin(\theta)$, $\theta = 0$ is normal incidence). Here $\mu_0 = 1$, $\varepsilon_0 = 1$, $\varepsilon_1 = 3$, $\alpha = 0.1$. The standard Brewster angle is noticeable. Here $\theta_{\rm B} = 60$ deg for model I (binary medium) and $\theta_{\rm B} \simeq 56$ deg for model II (uniform model).*

transverse wavector $\boldsymbol{\kappa}$ and frequency ω through the surface $z = z_0$ is given by the z-component of the Poynting vector $\hat{\boldsymbol{P}}(\omega,\boldsymbol{\kappa},z_0) = \frac{1}{2}\hat{\boldsymbol{E}} \times \overline{\hat{\boldsymbol{H}}}(\omega,\boldsymbol{\kappa},z_0)$:

$$\hat{P}_3(\omega,\boldsymbol{\kappa},z_0) = \frac{1}{2}\big(\hat{E}_1\overline{\hat{H}_2} - \hat{E}_2\overline{\hat{H}_1}\big)(\omega,\boldsymbol{\kappa},z_0) = \frac{1}{2}\big(\hat{E}_p\overline{\hat{H}_p} - \hat{E}_s\overline{\hat{H}_s}\big)(\omega,\boldsymbol{\kappa},z_0). \tag{4.18}$$

Using (4.5-4.12) and (3.24-3.27) we find that the incoming flux, reflected flux, and transmitted flux through the interface $z = 0$ are

$$\hat{P}_{\rm inc}(\omega,\boldsymbol{\kappa}) = \frac{1}{2}\Big(|\hat{a}_{\rm inc,p}(\omega,\boldsymbol{\kappa},0)|^2\frac{\kappa_0(\boldsymbol{\kappa})}{\varepsilon_0} + |\hat{a}_{\rm inc,s}(\omega,\boldsymbol{\kappa},0)|^2\frac{\mu_0}{\kappa_0(\boldsymbol{\kappa})}\Big),$$
$$\hat{P}_{\rm ref}(\omega,\boldsymbol{\kappa}) = -\frac{1}{2}\Big(|\hat{a}_{\rm inc,p}(\omega,\boldsymbol{\kappa},0)|^2|R^{\rm i}_{\rm p}(\boldsymbol{\kappa})|^2\frac{\kappa_0(\boldsymbol{\kappa})}{\varepsilon_0} + |\hat{a}_{\rm inc,s}(\omega,\boldsymbol{\kappa},0)|^2|R^{\rm i}_{\rm s}(\boldsymbol{\kappa})|^2\frac{\mu_0}{\kappa_0(\boldsymbol{\kappa})}\Big),$$
$$\hat{P}_{\rm tr}(\omega,\boldsymbol{\kappa}) = \frac{1}{2}\Big(|\hat{a}_{\rm inc,p}(\omega,\boldsymbol{\kappa},0)|^2|T^{\rm i}_{\rm p}(\boldsymbol{\kappa})|^2\frac{\kappa_p(\boldsymbol{\kappa})}{\varepsilon_{\rm a}} + |\hat{a}_{\rm inc,s}(\omega,\boldsymbol{\kappa},0)|^2|T^{\rm i}_{\rm s}(\boldsymbol{\kappa})|^2\frac{\mu_0}{\kappa_{\rm s}(\boldsymbol{\kappa})}\Big).$$

The conservation relation $\hat{P}_3(\omega,\boldsymbol{\kappa},z=0^-) = \hat{P}_3(\omega,\boldsymbol{\kappa},z=0^+)$ gives $\hat{P}_{\rm inc}(\omega,\boldsymbol{\kappa}) + \hat{P}_{\rm ref}(\omega,\boldsymbol{\kappa}) = \hat{P}_{\rm tr}(\omega,\boldsymbol{\kappa})$, which implies (4.16-4.17).

Since the incoming wave is propagating and comes from the homogeneous half-space $z < 0$ with permittivity ε_0, $|\boldsymbol{\kappa}|$ takes values in $[0, c_0^{-1})$. Therefore, provided $\varepsilon_{\rm a}$ and $\varepsilon_{\rm h}$ are larger than ε_0, $\kappa_{\rm p}$ and $\kappa_{\rm s}$ are always positive and there is never total reflection.

Note in particular that the p-reflection coefficient for the interface can vanish for a particular angle, called Brewster angle, such that (see Figure 4.1)

$$|\boldsymbol{\kappa}_{\rm B}|^2 c_0^2 = \frac{\varepsilon_{\rm a} - \varepsilon_0}{\varepsilon_{\rm a} - \varepsilon_0^2/\varepsilon_{\rm h}}, \qquad \theta_{\rm B} = \arcsin\big(|\boldsymbol{\kappa}_{\rm B}|c_0\big). \tag{4.19}$$

The Brewster angle depends on both the arithmetic and harmonic averages of the permittivity. Here, the angle θ is the angle in between in between the propagation axis z and the propagation direction.
When there is no fluctuation, i.e. when $\varepsilon_{\rm r}(z) \equiv \varepsilon_1$, then

$$|\boldsymbol{\kappa}_{\rm B}|^2 c_0^2 = \frac{\varepsilon_1}{\varepsilon_0 + \varepsilon_1},$$

and $\theta_{\rm B} = \arctan(\sqrt{\varepsilon_1/\varepsilon_0})$. This is a classical formula that predicts full transmission for the p-wave through an interface between two homogeneous half-spaces at the Brewster angle [3].
For model I, we have

$$|\boldsymbol{\kappa}_{\rm B}|^2 c_0^2 = \frac{\varepsilon_1}{\varepsilon_0+\varepsilon_1},$$

which is independent of the volume fraction α. Thus, when the incident slowness vector $\boldsymbol{\kappa}$ is such that we have full transmission through the interface from a medium with permittivity ε_0 to a medium with permittivity ε_1, then we also have full transmission for the same slowness vector through the interface from a medium with permittivity ε_1 to a medium with permittivity ε_0 by reciprocity. As a result we have full transmission through any binary medium made of material with permittivities ε_0 and ε_1.
For model II, we have

$$|\boldsymbol{\kappa}_{\rm B}|^2 c_0^2 = \frac{(\varepsilon_1-\varepsilon_0)^2}{\varepsilon_1^2-\varepsilon_0^2-2\varepsilon_0^2\log(\varepsilon_1/\varepsilon_0)}.$$

More generally when the fluctuations have weak amplitudes, of order η, we have $\varepsilon_{\rm h} = \varepsilon_{\rm a}+O(\eta^2)$ so that $|\boldsymbol{\kappa}_{\rm B}|^2c_0^2 = \varepsilon_{\rm a}/(\varepsilon_0+\varepsilon_{\rm a})+O(\eta^2)$ and $\theta_{\rm B} = \arctan(\sqrt{\varepsilon_{\rm a}/\varepsilon_0})+O(\eta^2)$.

5. Transmission in the diffusion approximation regime We now consider long propagation times, of the order of δ^{-1}. We then take $L^\delta = L/\delta$. We introduce the right-going modes $\hat{a}_q^\delta(\omega,\boldsymbol{\kappa},z)$, $q={\rm p,s}$, and left-going modes $\hat{b}_q^\delta(\omega,\boldsymbol{\kappa},z)$, $q={\rm p,s}$, defined in the region $z\in[0,L]$ by

$$\hat{H}_{\rm p}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big) = \hat{a}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z)e^{i\frac{\omega\kappa_{\rm p}z}{\delta}} + \hat{b}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z)e^{-i\frac{\omega\kappa_{\rm p}z}{\delta}}, \tag{5.1}$$

$$\hat{E}_{\rm p}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big) = \frac{\kappa_{\rm p}}{\varepsilon_{\rm a}}\big(\hat{a}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z)e^{i\frac{\omega\kappa_{\rm p}z}{\delta}} - \hat{b}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z)e^{-i\frac{\omega\kappa_{\rm p}z}{\delta}}\big), \tag{5.2}$$

$$\hat{H}_{\rm s}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big) = \hat{a}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z)e^{i\frac{\omega\kappa_{\rm s}z}{\delta}} + \hat{b}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z)e^{-i\frac{\omega\kappa_{\rm s}z}{\delta}}, \tag{5.3}$$

$$\hat{E}_{\rm s}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big) = \frac{\mu_0}{\kappa_{\rm s}}\big(-\hat{a}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z)e^{i\frac{\omega\kappa_{\rm s}z}{\delta}} + \hat{b}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z)e^{-i\frac{\omega\kappa_{\rm s}z}{\delta}}\big), \tag{5.4}$$

where $\kappa_{\rm p} = \kappa_{\rm p}(\boldsymbol{\kappa})$ and $\kappa_{\rm s} = \kappa_{\rm s}(\boldsymbol{\kappa})$ are given by (4.4). It follows for $z\in[0,L]$:

$$\hat{a}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z) = \frac{1}{2}e^{-i\frac{\omega\kappa_{\rm p}z}{\delta}}\Big(\hat{H}_{\rm p}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big) + \frac{\varepsilon_{\rm a}}{\kappa_{\rm p}}\hat{E}_{\rm p}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big)\Big), \tag{5.5}$$

$$\hat{b}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z) = \frac{1}{2}e^{i\frac{\omega\kappa_{\rm p}z}{\delta}}\Big(\hat{H}_{\rm p}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big) - \frac{\varepsilon_{\rm a}}{\kappa_{\rm p}}\hat{E}_{\rm p}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big)\Big), \tag{5.6}$$

$$\hat{a}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z) = \frac{1}{2}e^{-i\frac{\omega\kappa_{\rm s}z}{\delta}}\Big(\hat{H}_{\rm s}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big) - \frac{\kappa_{\rm s}}{\mu_0}\hat{E}_{\rm p}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big)\Big), \tag{5.7}$$

$$\hat{b}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z) = \frac{1}{2}e^{i\frac{\omega\kappa_{\rm s}z}{\delta}}\Big(\hat{H}_{\rm s}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big) + \frac{\kappa_{\rm s}}{\mu_0}\hat{E}_{\rm s}\big(\omega,\boldsymbol{\kappa},\frac{z}{\delta}\big)\Big). \tag{5.8}$$

The pair of p-mode amplitudes satisfies the system

$$\begin{aligned}
\partial_z \hat{a}_{\rm p}^\delta = & \frac{i\omega}{2}\Big[\kappa_{\rm p}\frac{1}{\delta}m(\frac{z}{\delta^2})-\frac{|\boldsymbol{\kappa}|^2}{\kappa_{\rm p}}\frac{1}{\delta}\tilde{m}(\frac{z}{\delta^2})+\Big(\kappa_{\rm p}+\frac{\varepsilon_{\rm a}^2|\boldsymbol{\kappa}|^2}{\varepsilon_{\rm r}^2(\frac{z}{\delta^2})\kappa_{\rm p}}\Big)\frac{1}{\delta^{1/2}}n(\frac{z}{\delta})\Big]\hat{a}_{\rm p}^\delta \\
& -\frac{i\omega}{2}\Big[\kappa_{\rm p}\frac{1}{\delta}m(\frac{z}{\delta^2})+\frac{|\boldsymbol{\kappa}|^2}{\kappa_{\rm p}}\frac{1}{\delta}\tilde{m}(\frac{z}{\delta^2})+\Big(\kappa_{\rm p}-\frac{\varepsilon_{\rm a}^2|\boldsymbol{\kappa}|^2}{\varepsilon_{\rm r}^2(\frac{z}{\delta^2})\kappa_{\rm p}}\Big)\frac{1}{\delta^{1/2}}n(\frac{z}{\delta})\Big]\hat{b}_{\rm p}^\delta e^{-2i\frac{\omega\kappa_{\rm p}z}{\delta}} \\
& -\frac{i\omega}{2}\frac{\varepsilon_{\rm a}|\boldsymbol{\kappa}|^2}{\varepsilon_{\rm r}^3(\frac{z}{\delta^2})\kappa_{\rm p}}\varepsilon_{\rm w}^2(\frac{z}{\delta})\hat{a}_{\rm p}^\delta-\frac{i\omega}{2}\frac{\varepsilon_{\rm a}|\boldsymbol{\kappa}|^2}{\varepsilon_{\rm r}^3(\frac{z}{\delta^2})\kappa_{\rm p}}\varepsilon_{\rm w}^2(\frac{z}{\delta})\hat{b}_{\rm p}^\delta e^{-2i\frac{\omega\kappa_{\rm p}z}{\delta}} \\
& -\frac{1}{2}\Big[\frac{\kappa_{\rm p}}{\varepsilon_{\rm a}}\varepsilon_{\rm d}(\frac{z}{\delta^2})+\frac{\varepsilon_{\rm a}|\boldsymbol{\kappa}|^2}{\kappa_{\rm p}}\frac{\varepsilon_{\rm d}}{\varepsilon_{\rm r}^2}(\frac{z}{\delta^2})\Big]\hat{a}_{\rm p}^\delta \\
& +\frac{1}{2}\Big[\frac{\kappa_{\rm p}}{\varepsilon_{\rm a}}\varepsilon_{\rm d}(\frac{z}{\delta^2})-\frac{\varepsilon_{\rm a}|\boldsymbol{\kappa}|^2}{\kappa_{\rm p}}\frac{\varepsilon_{\rm d}}{\varepsilon_{\rm r}^2}(\frac{z}{\delta^2})\Big]\hat{b}_{\rm p}^\delta e^{-2i\frac{\omega\kappa_{\rm p}z}{\delta}},
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
\partial_z \hat{b}_{\rm p}^\delta = & \frac{i\omega}{2}\Big[\kappa_{\rm p}\frac{1}{\delta}m(\frac{z}{\delta^2})+\frac{|\boldsymbol{\kappa}|^2}{\kappa_{\rm p}}\frac{1}{\delta}\tilde{m}(\frac{z}{\delta^2})+\Big(\kappa_{\rm p}-\frac{\varepsilon_{\rm a}^2|\boldsymbol{\kappa}|^2}{\varepsilon_{\rm r}^2(\frac{z}{\delta^2})\kappa_{\rm p}}\Big)\frac{1}{\delta^{1/2}}n(\frac{z}{\delta})\Big]\hat{a}_{\rm p}^\delta e^{2i\frac{\omega\kappa_{\rm p}z}{\delta}} \\
& -\frac{i\omega}{2}\Big[\kappa_{\rm p}\frac{1}{\delta}m(\frac{z}{\delta^2})-\frac{|\boldsymbol{\kappa}|^2}{\kappa_{\rm p}}\frac{1}{\delta}\tilde{m}(\frac{z}{\delta^2})+\Big(\kappa_{\rm p}+\frac{\varepsilon_{\rm a}^2|\boldsymbol{\kappa}|^2}{\varepsilon_{\rm r}^2(\frac{z}{\delta^2})\kappa_{\rm p}}\Big)\frac{1}{\delta^{1/2}}n(\frac{z}{\delta})\Big]\hat{b}_{\rm p}^\delta \\
& +\frac{i\omega}{2}\frac{\varepsilon_{\rm a}|\boldsymbol{\kappa}|^2}{\varepsilon_{\rm r}^3(\frac{z}{\delta^2})\kappa_{\rm p}}\varepsilon_{\rm w}^2(\frac{z}{\delta})\hat{a}_{\rm p}^\delta e^{2i\frac{\omega\kappa_{\rm p}z}{\delta}}+\frac{i\omega}{2}\frac{\varepsilon_{\rm a}|\boldsymbol{\kappa}|^2}{\varepsilon_{\rm r}^3(\frac{z}{\delta^2})\kappa_{\rm p}}\varepsilon_{\rm w}^2(\frac{z}{\delta})\hat{b}_{\rm p}^\delta \\
& -\frac{1}{2}\Big[\frac{\kappa_{\rm p}}{\varepsilon_{\rm a}}\varepsilon_{\rm d}(\frac{z}{\delta^2})-\frac{\varepsilon_{\rm a}|\boldsymbol{\kappa}|^2}{\kappa_{\rm p}}\frac{\varepsilon_{\rm d}}{\varepsilon_{\rm r}^2}(\frac{z}{\delta^2})\Big]\hat{a}_{\rm p}^\delta e^{2i\frac{\omega\kappa_{\rm p}z}{\delta}} \\
& +\frac{1}{2}\Big[\frac{\kappa_{\rm p}}{\varepsilon_{\rm a}}\varepsilon_{\rm d}(\frac{z}{\delta^2})+\frac{\varepsilon_{\rm a}|\boldsymbol{\kappa}|^2}{\kappa_{\rm p}}\frac{\varepsilon_{\rm d}}{\varepsilon_{\rm r}^2}(\frac{z}{\delta^2})\Big]\hat{b}_{\rm p}^\delta,
\end{aligned} \tag{5.10}$$

where we have neglected terms of order $O(\delta^{1/2})$ and we have denoted

$$m(z)=\frac{\varepsilon_{\rm r}(z)-\varepsilon_{\rm a}}{\varepsilon_{\rm a}}, \qquad \tilde{m}(z)=\frac{\varepsilon_{\rm a}}{\varepsilon_{\rm r}(z)}-\frac{\varepsilon_{\rm a}}{\varepsilon_{\rm h}}, \qquad n(z)=\frac{\varepsilon_{\rm w}(z)}{\varepsilon_{\rm a}}, \tag{5.11}$$

and we have assumed for simplicity that $\mathbb{E}[\varepsilon_{\rm w}(0)]=0$ (although we could address the general case $\mathbb{E}[\varepsilon_{\rm w}(0)]\neq 0$). Note that the three random processes $m(z)$, $\tilde{m}(z)$, and $n(z)$ have mean zero, and that the random processes $m(z)$ and $\tilde{m}(z)$ are correlated.

The pair of s-mode amplitudes satisfies the system

$$\begin{aligned}
\partial_z \hat{a}_{\rm s}^\delta = & \frac{i\omega}{2}\Big[\frac{\varepsilon_{\rm a}}{\varepsilon_0 c_0^2\kappa_{\rm s}}\frac{1}{\delta}m(\frac{z}{\delta^2})+\frac{\varepsilon_{\rm a}}{\varepsilon_0 c_0^2\kappa_{\rm s}}\frac{1}{\delta^{1/2}}n(\frac{z}{\delta})\Big]\hat{a}_{\rm s}^\delta \\
& -\frac{i\omega}{2}\Big[\frac{\varepsilon_{\rm a}}{\varepsilon_0 c_0^2\kappa_{\rm s}}\frac{1}{\delta}m(\frac{z}{\delta^2})+\frac{\varepsilon_{\rm a}}{\varepsilon_0 c_0^2\kappa_{\rm s}}\frac{1}{\delta^{1/2}}n(\frac{z}{\delta})\Big]\hat{b}_{\rm s}^\delta e^{-2i\frac{\omega\kappa_{\rm s}z}{\delta}} \\
& -\frac{1}{2}\Big[\frac{1}{\varepsilon_0 c_0^2\kappa_{\rm s}}\varepsilon_{\rm d}(\frac{z}{\delta^2})\Big]\hat{a}_{\rm s}^\delta+\frac{1}{2}\Big[\frac{1}{\varepsilon_0 c_0^2\kappa_{\rm s}}\varepsilon_{\rm d}(\frac{z}{\delta^2})\Big]\hat{b}_{\rm s}^\delta e^{-2i\frac{\omega\kappa_{\rm s}z}{\delta}},
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\partial_z \hat{b}_{\rm s}^\delta = & \frac{i\omega}{2}\Big[\frac{\varepsilon_{\rm a}}{\varepsilon_0 c_0^2\kappa_{\rm s}}\frac{1}{\delta}m(\frac{z}{\delta^2})+\frac{\varepsilon_{\rm a}}{\varepsilon_0 c_0^2\kappa_{\rm s}}\frac{1}{\delta^{1/2}}n(\frac{z}{\delta})\Big]\hat{a}_{\rm s}^\delta e^{2i\frac{\omega\kappa_{\rm s}z}{\delta}} \\
& -\frac{i\omega}{2}\Big[\frac{\varepsilon_{\rm a}}{\varepsilon_0 c_0^2\kappa_{\rm s}}\frac{1}{\delta}m(\frac{z}{\delta^2})+\frac{\varepsilon_{\rm a}}{\varepsilon_0 c_0^2\kappa_{\rm s}}\frac{1}{\delta^{1/2}}n(\frac{z}{\delta})\Big]\hat{b}_{\rm s}^\delta \\
& -\frac{1}{2}\Big[\frac{1}{\varepsilon_0 c_0^2\kappa_{\rm s}}\varepsilon_{\rm d}(\frac{z}{\delta^2})\Big]\hat{a}_{\rm s}^\delta e^{2i\frac{\omega\kappa_{\rm s}z}{\delta}}+\frac{1}{2}\Big[\frac{1}{\varepsilon_0 c_0^2\kappa_{\rm s}}\varepsilon_{\rm d}(\frac{z}{\delta^2})\Big]\hat{b}_{\rm s}^\delta.
\end{aligned} \tag{5.13}$$

These two systems are complemented by boundary conditions corresponding to an incoming wave from the left (conditions (3.22-3.23)) and radiation conditions from the right (conditions (3.30-3.31)) that read in terms of the mode amplitudes:

$$\hat{a}_q^\delta(\omega,\boldsymbol{\kappa},z=0)=\hat{a}_{\mathrm{inc},q}(\omega,\boldsymbol{\kappa}),\qquad \hat{b}_q^\delta(\omega,\boldsymbol{\kappa},z=L)=0,\qquad q=\mathrm{p},\mathrm{s},\tag{5.14}$$

where $\hat{a}_{\mathrm{inc},q}$, $q=\mathrm{p},\mathrm{s}$, are given by (3.28-3.29).

5.1. Weak convergence of the transmitted pulse Let us assume that the incident wave is a broadband p- or s-polarized plane wave with the incident wavevector along the direction $(c_0\boldsymbol{\kappa}_{\mathrm{inc}},\sqrt{1-c_0^2|\boldsymbol{\kappa}_{\mathrm{inc}}|^2})$. The amplitudes of the incoming waves are

$$a_{\mathrm{inc},q}(t,\boldsymbol{x})=f_q(t-\boldsymbol{\kappa}_{\mathrm{inc}}\cdot\boldsymbol{x}),\qquad \hat{a}_{\mathrm{inc},q}(\omega,\boldsymbol{\kappa})=\frac{(2\pi)^2}{\omega^2}\hat{f}_q(\omega)\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}_{\mathrm{inc}}),\tag{5.15}$$

for q=p or s, where f_q is some given pulse profile. In the limit $\delta\to 0$, following [6, Chapter 8] we find that the wave transmitted at z/δ and observed around its expected arrival time $\kappa_q(\boldsymbol{\kappa}_{\mathrm{inc}})z/\delta$ has the form

$$a_q\Big(\frac{\kappa_q(\boldsymbol{\kappa}_{\mathrm{inc}})z}{\delta}+t,\boldsymbol{x},\frac{z}{\delta}\Big)\overset{\delta\to 0}{\longrightarrow}K_{q,\boldsymbol{\kappa}_{\mathrm{inc}},z}*f_q\big(t-\boldsymbol{\kappa}_{\mathrm{inc}}\cdot\boldsymbol{x}-\tau_{q,\boldsymbol{\kappa}_{\mathrm{inc}},z}\big),\qquad q=\mathrm{p},\mathrm{s}.\tag{5.16}$$

Here $t\to K_{q,\boldsymbol{\kappa}_{\mathrm{inc}},z}(t)$ is a deterministic convolution kernel and $\tau_{q,\boldsymbol{\kappa}_{\mathrm{inc}},z}$ is a random variable. This means that the transmitted wave undergoes both a deterministic deformation and a random travel time correction. More precisely, the Fourier transform of the deterministic convolution kernel is given by

$$\hat{K}_{q,\boldsymbol{\kappa}_{\mathrm{inc}},z}(\omega)=T_q^{\mathrm{i}}(\boldsymbol{\kappa}_{\mathrm{inc}})\hat{T}_q(\omega,\boldsymbol{\kappa}_{\mathrm{inc}},z),\tag{5.17}$$

$$\hat{T}_q(\omega,\boldsymbol{\kappa},z)=\exp\Big(-\frac{\gamma_q(\omega,\boldsymbol{\kappa})\omega^2 z}{8}-\frac{1}{2}\sigma_q(\boldsymbol{\kappa})z\Big),\qquad q=\mathrm{p},\mathrm{s}.\tag{5.18}$$

The convolution kernel contains the term $T_q^{\mathrm{i}}(\boldsymbol{\kappa}_{\mathrm{inc}})$ that is the transmission coefficient (4.15) of the interface $z=0$ and the term $\hat{T}_q(\omega,\boldsymbol{\kappa}_{\mathrm{inc}},z)$ that is the transmission coefficient of the slab $[0,z]$ of random medium. The term $\hat{T}_q(\omega,\boldsymbol{\kappa}_{\mathrm{inc}},z)$ consists of effective dissipation and effective diffusion/dispersion.
The random travel time corrections $\tau_{q,\boldsymbol{\kappa}_{\mathrm{inc}},z}$ are random and zero-mean, and their statistics are described below. Although the p- and s-waves propagate without interaction in this regime, the random travel time corrections for the two waves are correlated.

The effective dissipation coefficient of the p-wave is given by

$$\sigma_{\mathrm{p}}(\boldsymbol{\kappa})=\kappa_{\mathrm{p}}(\boldsymbol{\kappa})\frac{\mathbb{E}[\varepsilon_{\mathrm{d}}(0)]}{\varepsilon_{\mathrm{a}}}+\varepsilon_{\mathrm{a}}\mathbb{E}[\varepsilon_{\mathrm{d}}(0)\varepsilon_{\mathrm{r}}(0)^{-2}]\frac{|\boldsymbol{\kappa}|^2}{\kappa_{\mathrm{p}}(\boldsymbol{\kappa})}.\tag{5.19}$$

The diffusion/dispersion coefficient of the p-wave is given by

$$\begin{aligned}\gamma_{\mathrm{p}}(\omega,\boldsymbol{\kappa})=2\int_0^\infty\mathbb{E}\Big[\big(\kappa_{\mathrm{p}}(\boldsymbol{\kappa})m(0)+\frac{|\boldsymbol{\kappa}|^2}{\kappa_{\mathrm{p}}(\boldsymbol{\kappa})}\tilde{m}(0)\big)\big(\kappa_{\mathrm{p}}(\boldsymbol{\kappa})m(\zeta)+\frac{|\boldsymbol{\kappa}|^2}{\kappa_{\mathrm{p}}(\boldsymbol{\kappa})}\tilde{m}(\zeta)\big)\Big]d\zeta\\ +2\Big(\kappa_{\mathrm{p}}(\boldsymbol{\kappa})-\frac{q_2|\boldsymbol{\kappa}|^2}{\kappa_{\mathrm{p}}(\boldsymbol{\kappa})}\Big)^2\int_0^\infty\mathbb{E}[n(0)n(\zeta)]e^{2i\omega\kappa_{\mathrm{p}}(\boldsymbol{\kappa})\zeta}d\zeta.\end{aligned}\tag{5.20}$$

In order to minimize the damping of the pulse due to the random medium the probing angle, as determined by $\boldsymbol{\kappa}$, should be chosen to minimize (the real part of) γ_{p} in (5.20). The first term and the real part of the second term in (5.20) are indeed non-negative by Bochner's theorem. In sections 5.2 and 5.3 we consider explicitly the diffusion coefficient and its minimization in the context of the particular models introduced above.

Next, observe that the random travel time correction of the p-wave is

$$\begin{aligned}\tau_{\mathrm{p},\boldsymbol{\kappa},z} = &\frac{1}{4}\Big[\big(\kappa_\mathrm{p}\sqrt{\gamma_m} - \frac{|\boldsymbol{\kappa}|^2}{\kappa_\mathrm{p}}\sqrt{\gamma_{\tilde m}}\big)\sqrt{1+\rho_{m\tilde m}} + \big(\kappa_\mathrm{p}\sqrt{\gamma_m} + \frac{|\boldsymbol{\kappa}|^2}{\kappa_\mathrm{p}}\sqrt{\gamma_{\tilde m}}\big)\sqrt{1-\rho_{m\tilde m}}\Big] W_1(z)\\ &+\frac{1}{4}\Big[\big(\kappa_\mathrm{p}\sqrt{\gamma_m} - \frac{|\boldsymbol{\kappa}|^2}{\kappa_\mathrm{p}}\sqrt{\gamma_{\tilde m}}\big)\sqrt{1+\rho_{m\tilde m}} - \big(\kappa_\mathrm{p}\sqrt{\gamma_m} + \frac{|\boldsymbol{\kappa}|^2}{\kappa_\mathrm{p}}\sqrt{\gamma_{\tilde m}}\big)\sqrt{1-\rho_{m\tilde m}}\Big] W_2(z)\\ &+\frac{1}{2}\big(\kappa_\mathrm{p} + \frac{q_2|\boldsymbol{\kappa}|^2}{\kappa_\mathrm{p}}\big)\sqrt{\gamma_n} W_3(z) - \frac{1}{2}\frac{q_0|\boldsymbol{\kappa}|^2}{\kappa_\mathrm{p}} z. \end{aligned} \tag{5.21}$$

The W_j, $j=1,2,3$, are independent Brownian motions. The coefficients $\gamma_\cdot$ are given in terms of the power spectral densities of the fluctuations of the medium:

$$\gamma_n = 2\int_0^\infty \mathbb{E}[n(0)n(\zeta)]d\zeta, \tag{5.22}$$

$$\gamma_m = 2\int_0^\infty \mathbb{E}[m(0)m(\zeta)]d\zeta, \tag{5.23}$$

$$\gamma_{\tilde m} = 2\int_0^\infty \mathbb{E}[\tilde m(0)\tilde m(\zeta)]d\zeta, \tag{5.24}$$

$$\rho_{m\tilde m} = \frac{\int_0^\infty \mathbb{E}[m(0)\tilde m(\zeta)]d\zeta + \int_0^\infty \mathbb{E}[\tilde m(0)m(\zeta)]d\zeta}{\sqrt{\gamma_m\gamma_{\tilde m}}}. \tag{5.25}$$

It follows that the random travel time correction τ is a Gaussian random variable with mean $-\frac{1}{2}\frac{q_0|\boldsymbol{\kappa}|^2}{\kappa_\mathrm{p}(\boldsymbol{\kappa})}z$ and variance

$$\mathrm{Var}\big(\tau_{\mathrm{p},\boldsymbol{\kappa},z}\big) = \frac{1}{4}\Big(\kappa_\mathrm{p}^2(\boldsymbol{\kappa})\gamma_m + \frac{|\boldsymbol{\kappa}|^4}{\kappa_\mathrm{p}^2(\boldsymbol{\kappa})}\gamma_{\tilde m} - 2|\boldsymbol{\kappa}|^2\rho_{m\tilde m}\sqrt{\gamma_m\gamma_{\tilde m}} + \big(\kappa_\mathrm{p}(\boldsymbol{\kappa}) + \frac{q_2|\boldsymbol{\kappa}|^2}{\kappa_\mathrm{p}(\boldsymbol{\kappa})}\big)^2\gamma_n\Big)z.$$

Here we have denoted

$$q_2 = \mathbb{E}[\varepsilon_\mathrm{r}(0)^{-2}]\varepsilon_\mathrm{a}^2, \qquad q_0 = \varepsilon_\mathrm{a}\mathbb{E}[\varepsilon_\mathrm{w}(0)^2]\mathbb{E}[\varepsilon_\mathrm{r}(0)^{-3}]. \tag{5.26}$$

The effective dissipation coefficient of the s-wave is given by

$$\sigma_\mathrm{s}(\boldsymbol{\kappa}) = \frac{1}{c_0^2\kappa_\mathrm{s}(\boldsymbol{\kappa})}\frac{\mathbb{E}[\varepsilon_\mathrm{d}(0)]}{\varepsilon_0}. \tag{5.27}$$

The diffusion/dispersion coefficient is given by

$$\gamma_\mathrm{s}(\omega,\boldsymbol{\kappa}) = \frac{2\varepsilon_\mathrm{a}^2}{\varepsilon_0^2c_0^4\kappa_\mathrm{s}^2(\boldsymbol{\kappa})}\int_0^\infty \mathbb{E}[m(0)m(\zeta)]d\zeta + \frac{2\varepsilon_\mathrm{a}^2}{\varepsilon_0^2c_0^4\kappa_\mathrm{s}^2(\boldsymbol{\kappa})}\int_0^\infty \mathbb{E}[n(0)n(\zeta)]e^{2i\omega\kappa_\mathrm{s}(\boldsymbol{\kappa})\zeta}d\zeta. \tag{5.28}$$

Observe again that the first term and the real part of the second term in this expression are non-negative by Bochner's theorem. Moreover, in this case the form of $\kappa_\mathrm{s}(\boldsymbol{\kappa})$ gives that the s-polarized component experiences least damping for normal incidence and that the damping increases with the angle $\arcsin(c_0|\boldsymbol{\kappa}|)$.

The random travel time correction is

$$\tau_{\mathrm{s},\boldsymbol{\kappa},z} = \frac{1}{4}\frac{\varepsilon_{\mathrm{a}}}{\varepsilon_0 c_0^2 \kappa_{\mathrm{s}}}\sqrt{\gamma_m}\Big[\sqrt{1+\rho_{m\tilde{m}}}+\sqrt{1-\rho_{m\tilde{m}}}\Big]W_1(z)$$
$$+\frac{1}{4}\frac{\varepsilon_{\mathrm{a}}}{\varepsilon_0 c_0^2 \kappa_{\mathrm{s}}}\sqrt{\gamma_m}\Big[\sqrt{1+\rho_{m\tilde{m}}}-\sqrt{1-\rho_{m\tilde{m}}}\Big]W_2(z)$$
$$+\frac{1}{2}\frac{\varepsilon_{\mathrm{a}}}{\varepsilon_0 c_0^2 \kappa_{\mathrm{s}}}\sqrt{\gamma_n}W_3(z). \tag{5.29}$$

It is a Gaussian random variable with mean zero and variance

$$\mathbb{E}[\tau_{\mathrm{s},\boldsymbol{\kappa},z}^2] = \frac{1}{4}\frac{\varepsilon_{\mathrm{a}}^2}{\varepsilon_0^2 c_0^4 \kappa_{\mathrm{s}}^2(\boldsymbol{\kappa})}(\gamma_m+\gamma_n)z. \tag{5.30}$$

Note that the Brownian motions are the same in (5.21) and in (5.29). The reciprocal of $\sigma_q(\boldsymbol{\kappa})$ gives the dissipation length for q-waves, $q=\mathrm{p},\mathrm{s}$:

$$L_{\mathrm{dis},q}(\omega,\boldsymbol{\kappa}) = \frac{2}{\sigma_q(\boldsymbol{\kappa})}, \qquad q=\mathrm{p},\mathrm{s}. \tag{5.31}$$

We remark that in original coordinates this length is scaled by δ^{-1}. The reciprocal of the real part of $\gamma_q(\omega,\boldsymbol{\kappa})$ gives the localization length for q-waves, $q=\mathrm{p},\mathrm{s}$:

$$L_{\mathrm{loc},q}(\omega,\boldsymbol{\kappa}) = \frac{8}{\omega^2 \mathrm{Re}(\gamma_q(\omega,\boldsymbol{\kappa}))}, \qquad q=\mathrm{p},\mathrm{s}. \tag{5.32}$$

It has been studied in detail in the literature for scalar waves [2, 11, 12] and for vector electromagnetic waves [1, 10, 13, 14, 17]. It turns out (see also Figure 5.1) that the localization length for the p-wave can take very large values or even diverge for a particular value of the angle, that we may call generalized Brewster angle θ_{gB}:

$$|\boldsymbol{\kappa}_{\mathrm{gB}}| = \underset{|\boldsymbol{\kappa}|}{\mathrm{argmin}}\,\mathrm{Re}(\gamma_{\mathrm{p}}(\omega,\boldsymbol{\kappa})), \qquad \theta_{\mathrm{gB}} = \arcsin(c_0|\boldsymbol{\kappa}_{\mathrm{gB}}|). \tag{5.33}$$

This generalized Brewster angle depends in general on the frequency ω, because the diffusion coefficient $\mathrm{Re}(\gamma_{\mathrm{p}}(\omega,\boldsymbol{\kappa}))$ depends in general on ω (see (5.20)).

In the following two subsections we discuss the transmission problem in the absence of dissipation. As shown by the expression of the bulk transmission coefficients $\hat{T}_q(\omega,\boldsymbol{\kappa},z)$, $q=\mathrm{p},\mathrm{s}$, dissipation gives rise to an additional damping effect that is weakly dependent on the incidence angle (see also Figure 5.2).

5.2. Discussion in the absence of slow and small fluctuations Note that $\gamma_q(\omega,\boldsymbol{\kappa})$, $q=\mathrm{p},\mathrm{s}$, does not depend on ω and it is nonnegative-real valued when $n(z)\equiv 0$:

$$\gamma_{\mathrm{p}}(\omega,\boldsymbol{\kappa}) \equiv \gamma_{\mathrm{p}}(\boldsymbol{\kappa}) = \kappa_{\mathrm{p}}^2(\boldsymbol{\kappa})\gamma_m + 2|\boldsymbol{\kappa}|^2\rho_{m\tilde{m}}\sqrt{\gamma_m\gamma_{\tilde{m}}} + \frac{|\boldsymbol{\kappa}|^4}{\kappa_{\mathrm{p}}^2(\boldsymbol{\kappa})}\gamma_{\tilde{m}}, \tag{5.34}$$

$$\gamma_{\mathrm{s}}(\omega,\boldsymbol{\kappa}) \equiv \gamma_{\mathrm{s}}(\boldsymbol{\kappa}) = \frac{\varepsilon_{\mathrm{a}}^2}{\varepsilon_0^2 c_0^4 \kappa_{\mathrm{s}}^2(\boldsymbol{\kappa})}\gamma_m. \tag{5.35}$$

The generalized Brewster angle θ_{gB} may or may not be equal to the standard Brewster angle θ_{B} for which the p-reflection coefficient of the interface $z=0$ vanishes, as shown by inspection of the two models.

model I: We have

$$\begin{aligned}\gamma_m &= 2l_c\alpha(1-\alpha)\frac{(\varepsilon_1-\varepsilon_0)^2}{\varepsilon_a^2},\\ \gamma_{\tilde m} &= 2l_c\alpha(1-\alpha)\frac{\varepsilon_a^2(\varepsilon_1-\varepsilon_0)^2}{\varepsilon_0^2\varepsilon_1^2},\\ \rho_{m\tilde m} &= -1.\end{aligned}$$

It can then be shown that $|\boldsymbol{\kappa}_{gB}|=|\boldsymbol{\kappa}_B|$ or equivalently $\theta_{gB}=\theta_B$. Note that the localization length at Brewster angle is infinite. There is indeed perfect transmission in this case, both the interface and bulk transmittivities are equal to one (this could be anticipated from the discussion at the end of Section 4). Note also that the Brewster angle does not depend on the volume fraction of material 1, nor on the correlation length of the medium. As shown in [17], the Brewster angle and generalized Brewster angle are equal only in this particular situation, a binary medium with one component permittivity equal to the one of the homogeneous half-space. It can be seen from (5.9) that perfect anticorrelation of the processes m and $\tilde m$ in addition to choosing the angle correctly to match the magnitude of their multiplicative factors cancel their contribution to the diffusion coefficient.

model II: We have

$$\begin{aligned}\gamma_m &= \frac{2l_c}{3}\frac{(\varepsilon_1-\varepsilon_0)^2}{(\varepsilon_1+\varepsilon_0)^2},\\ \gamma_{\tilde m} &= 2l_c\Big(\frac{\varepsilon_a^2}{\varepsilon_0\varepsilon_1}-\frac{\varepsilon_a^2}{\varepsilon_h^2}\Big),\\ \rho_{m\tilde m} &= \frac{2l_c(1-\varepsilon_a/\varepsilon_h)}{\sqrt{\gamma_m\gamma_{\tilde m}}},\end{aligned}$$

and therefore $|\boldsymbol{\kappa}_{gB}|\neq|\boldsymbol{\kappa}_B|$. In this case (which is the general case) one cannot maximize both the interface transmittivity and the bulk transmittivity. If one wants to get maximal transmission, then a trade-off has to be made in order to maximize (with respect to $\boldsymbol{\kappa}$) the deterministic product $T_p^i(\boldsymbol{\kappa})\hat T_p(\omega,\boldsymbol{\kappa},z)$. If the propagation depth z is not large (smaller than the localization length), then the optimal angle is close to the standard Brewster angle θ_B maximizing $T_p^i(\boldsymbol{\kappa})$. If the propagation depth z is large, then the optimal angle is close to the generalized Brewster angle θ_{gB} maximizing $\hat T_p(\omega,\boldsymbol{\kappa},z)$ or equivalently minimizing $\gamma_p(\boldsymbol{\kappa})$. The localization length for the two models are shown in Figure 5.1 for some specific values for the parameters in the case of absence of slow and small fluctuations. We may refer to this effect as an apparent diffusion or attenuation effect since it is caused by the medium fluctuations giving scattering rather than being caused by intrinsic attenuation or loss. The intrinsic or effective attenuation, σ_p in (5.19), is shown in Figure 5.2 for some specific values of the parameters.

5.3. Discussion in the presence of slow and small fluctuations The presence of slow and small fluctuations can play a critical role. Indeed it will not affect the homogenized parameters, and therefore it does not modify the value of the standard Brewster angle, but it may strongly modify the generalized Brewster angle that corresponds to the maximal localization length. If the small and slow fluctuations dominate so that

$$\mathrm{Re}(\gamma_p(\omega,\boldsymbol{\kappa}))=2\Big(\kappa_p(\boldsymbol{\kappa})-\frac{q_2|\boldsymbol{\kappa}|^2}{\kappa_p(\boldsymbol{\kappa})}\Big)^2\int_0^\infty \mathbb{E}\big[n(0)n(\zeta)\big]\cos\big(2\omega\kappa_p(\boldsymbol{\kappa})\zeta\big)d\zeta, \qquad (5.36)$$

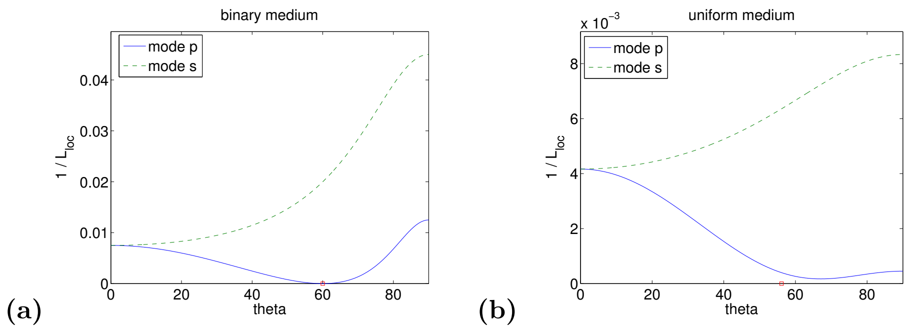


FIG. 5.1. *Localization lengths, $L_{\rm loc}, q$, as determined by (5.32), (5.34) and (5.35) for the two models and the p,s-waves. Here $\mu_0=1$, $\varepsilon_0=1$, $\varepsilon_1=3$, $\alpha=0.1$, $l_{\rm c}=0.1$, $\omega=1$ and the absence of small and slow fluctuations. The generalized Brewster angle is noticeable, which is equal to, resp. different from, the standard Brewster angle in the case of model I (binary medium), resp model II (uniform medium). Here $\theta_{\rm gB}=60$ deg for model I (binary medium) and $\theta_{\rm gB}\simeq 67$ deg for model II (uniform model). Note that for model II we have $L_{\rm loc,p}(\boldsymbol{\kappa}_{\rm gB})\simeq 5800$ while $L_{\rm loc,p}(\boldsymbol{\kappa}_{\rm B})\simeq 2500$, which means that the bulk transmittivity can be very different at the two angles. The Brewster angle is shown by the $*$ on the horizontal axis.*

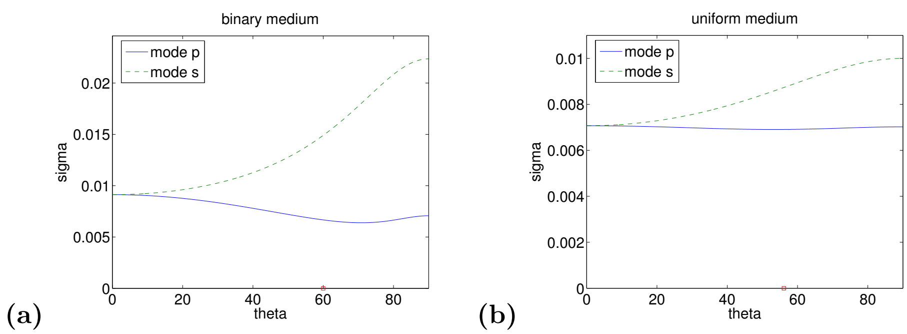


FIG. 5.2. *Effective dissipations for the two models and the p,s-waves. Here $\mu_0=1$, $\varepsilon_0=1$, $\varepsilon_1=3$, $\bar{\varepsilon}_{\rm d}=0.01$, $\alpha=0.1$. In model I (binary medium), dissipation is present only in the material 1 and is equal to $\bar{\varepsilon}_{\rm d}$: $\varepsilon_{\rm d}(z)=\bar{\varepsilon}_{\rm d}\mathbf{1}_{\varepsilon(z)=\varepsilon_1}$. In model II (uniform medium), dissipation is proportional to $\varepsilon_{\rm r}(z)-\varepsilon_0$: $\varepsilon_d(z)=\bar{\varepsilon}_{\rm d}(\varepsilon_{\rm r}(z)-\varepsilon_0)/(\varepsilon_1-\varepsilon_0)$.*

then the generalized Brewster angle is given by

$$c_0^2|\boldsymbol{\kappa}_{\rm gB}|^2=\frac{\varepsilon_{\rm a}\varepsilon_{\rm h}}{\varepsilon_{\rm a}\varepsilon_0+q_2\varepsilon_{\rm h}\varepsilon_0}=\frac{\varepsilon_{\rm h}}{\varepsilon_0}\frac{1}{1+\varepsilon_{\rm a}\varepsilon_{\rm h}\mathbb{E}[\varepsilon_{\rm r}(0)^{-2}]}. \tag{5.37}$$

At the generalized Brewster angle the localization length diverges.

In particular, if $\varepsilon_{\rm r}(z)$ is approximately constant and equal to ε_1 (this is model I with $\alpha\approx 1$), then $c_0^2|\boldsymbol{\kappa}_{\rm gB}|^2=\varepsilon_1/(2\varepsilon_0)$ to leading order, while $c_0^2|\boldsymbol{\kappa}_{\rm B}|^2=\varepsilon_1/(\varepsilon_0+\varepsilon_1)$. Again the standard and generalized Brewster angles can be quite different, which means that it is not possible to achieve perfect transmission both at the interface and in the bulk medium.

In general, when neither the large and rapid fluctuations nor the small and slow fluctuations dominate, the diffusion coefficient $\mathrm{Re}(\gamma_q(\omega,\boldsymbol{\kappa}))$ defined by (5.20) is the sum of the terms (5.34) and (5.36). These two terms are nonnegative and they may vanish, but at different wavevectors/angles. Therefore the gobal minimum does not correspond to a divergence of the localization length (or full transmission) in general, but to a large value (but not infinite) of the localization length.

6. Reflection in the diffusion approximation regime In this section we study the statistics of the reflected wave. We again consider times of order δ^{-1} and we take again $L^\delta = L/\delta$. We consider an incoming plane wave of the form (5.15) that is a broadband p- or s-polarized plane wave with the incident wavevector along the direction $(c_0\boldsymbol{\kappa}_{\rm inc}, \sqrt{1-c_0^2|\boldsymbol{\kappa}_{\rm inc}|^2})$. The analysis follows the one developed in the context of acoustic waves in [6, Section 17.1.3].

The reflected wave (at the surface $z=0^-$) has the form

$$H_q(t,\boldsymbol{x}) = H_{{\rm ref},q}(t-\boldsymbol{\kappa}_{\rm inc}\cdot\boldsymbol{x}), \tag{6.1}$$

$$H_{{\rm ref},q}(t) = \frac{1}{2\pi}\int \hat{\mathcal{R}}_q^\delta(\omega,\boldsymbol{\kappa}_{\rm inc})\hat{f}_q(\omega)e^{-i\omega t}d\omega, \tag{6.2}$$

where $\hat{\mathcal{R}}_q^\delta(\omega,\boldsymbol{\kappa})$ is the reflection coefficient taking into account both the reflection from the interface and from the bulk:

$$\hat{\mathcal{R}}_q^\delta(\omega,\boldsymbol{\kappa}) = \frac{R_q^{\rm i}(\boldsymbol{\kappa}) + \hat{R}_q^\delta(\omega,\boldsymbol{\kappa},L)}{1+R_q^{\rm i}(\boldsymbol{\kappa})\hat{R}_q^\delta(\omega,\boldsymbol{\kappa},L)},$$

which can also be written as

$$\hat{\mathcal{R}}_q^\delta(\omega,\boldsymbol{\kappa}) = R_q^{\rm i}(\boldsymbol{\kappa}) + \frac{(1-R_q^{\rm i}(\boldsymbol{\kappa})^2)\hat{R}_q^\delta(\omega,\boldsymbol{\kappa},L)}{1+R_q^{\rm i}(\boldsymbol{\kappa})\hat{R}_q^\delta(\omega,\boldsymbol{\kappa},L)}. \tag{6.3}$$

Here $R_q^{\rm i}(\boldsymbol{\kappa})$ is the reflection coefficient (4.13) of the interface $z=0$ and $\hat{R}_q^\delta(\omega,\boldsymbol{\kappa},L)$ is the reflection coefficient of the slab $[0,L]$ of random medium. It can be defined as $\hat{R}_q^\delta(\omega,\boldsymbol{\kappa},z) = \hat{b}_q^\delta(\omega,\boldsymbol{\kappa},z)/\hat{a}_q^\delta(\omega,\boldsymbol{\kappa},z)$, where $(\hat{a}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z),\hat{b}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z))$ satisfies the system (5.9-5.10) starting from $\hat{a}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z=0)=1$ and $\hat{b}_{\rm p}^\delta(\omega,\boldsymbol{\kappa},z=0)=0$ and $(\hat{a}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z),\hat{b}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z))$ satisfies the system (5.12-5.13) starting from $\hat{a}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z=0)=1$ and $\hat{b}_{\rm s}^\delta(\omega,\boldsymbol{\kappa},z=0)=0$. As a result $\hat{R}_q^\delta(\omega,\boldsymbol{\kappa},z)$ satisfies a closed-form Riccati equation starting from $\hat{R}_q^\delta(\omega,\boldsymbol{\kappa},z=0)=0$. This equation has been studied in detail in [6, Chapter 14] and the limiting moments of $\hat{R}_q^\delta(\omega,\boldsymbol{\kappa},z)$ as $\delta\to 0$ have been obtained. We will use these results below.

The reflected wave consists of a coherent wave that is reflected immediately by the interface

$$H_{{\rm ref},q}(t) = R_q^{\rm i}(\boldsymbol{\kappa}_{\rm inc})f_q(t) \tag{6.4}$$

and of small (ie of typical amplitude $\delta^{1/2}$) incoherent (ie zero-mean) waves which have penetrated the bulk medium before being reflected and whose spectrum is locally stationary and of the following form in the asymptotic $\delta\to 0$ (for $\tau>0$):

$$\frac{1}{\delta}\mathbb{E}\Big[H_{{\rm ref},q}\big(\frac{\tau}{\delta}\big)H_{{\rm ref},q}\big(\frac{\tau}{\delta}+t\big)\Big] \overset{\delta\to 0}{\longrightarrow} \frac{1}{(2\pi)^2}\iint \mathcal{U}_q(\omega,\boldsymbol{\kappa}_{\rm inc},h)|\hat{f}_q(\omega)|^2 e^{ih\tau+i\omega t}d\omega dh, \tag{6.5}$$

where

$$\mathcal{U}_q(\omega,\boldsymbol{\kappa},h) = \lim_{\delta\to 0}\mathbb{E}\big[\hat{\mathcal{R}}_q^\delta(\omega,\boldsymbol{\kappa})\overline{\hat{\mathcal{R}}_q^\delta(\omega+\delta h,\boldsymbol{\kappa})}\big]. \tag{6.6}$$

This limit can be obtained following the procedure described in [6, Section 17.1.3]:

$$\mathcal{U}_q(\omega,\boldsymbol{\kappa},h) = R^{\rm i}_q(\boldsymbol{\kappa})^2 + \sum_{n=0}^{\infty} (1-R^{\rm i}_q(\boldsymbol{\kappa})^2)^2 R^{\rm i}_q(\boldsymbol{\kappa})^{2n} \int_0^\infty \mathcal{W}_{q,n+1}(\omega,\boldsymbol{\kappa},s) e^{-ishc_0^2\kappa_q^2(\boldsymbol{\kappa})} ds. \tag{6.7}$$

The first component $R^{\rm i}_q(\boldsymbol{\kappa})^2$ actually corresponds to the coherent wave (6.4) emerging at $\tau=0$. The other components (in the sum over n) correspond to waves that have penetrated into the random medium and that have been reflected back and forth between the interface and the bulk n times before emerging into the left homogeneous half-space.

The functions $\mathcal{W}_{q,n}(\omega,\boldsymbol{\kappa},s) = W_{q,n}(\omega,\boldsymbol{\kappa},s,L)$ where the $(W_{q,n}(\omega,\boldsymbol{\kappa},s,z))_{z\in[0,L]}$ are solutions of the following system of transport equations:

$$\begin{aligned}\frac{\partial W_{q,n}}{\partial z} + \frac{2n}{c_0^2\kappa_q(\boldsymbol{\kappa})}\frac{\partial W_{q,n}}{\partial s} = & \frac{{\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}))\omega^2 n^2}{4}\big(W_{q,n+1}+W_{q,n-1}-2W_{q,n}\big) \\ & -2n\sigma_q(\boldsymbol{\kappa})W_{q,n},\end{aligned} \tag{6.8}$$

starting from $W_{q,n}(\omega,\boldsymbol{\kappa},s,z=0) = \delta(s)\mathbf{1}_0(n)$. We can take the limit $L\to\infty$ in this system as we know that our quantities of interest do not depend on L. Then the functions $\mathcal{W}_{q,n}$ can be identified as the solutions of the following stationary system

$$\begin{aligned}\frac{\partial \mathcal{W}_{q,n}}{\partial s} = & \frac{c_0^2\kappa_q(\boldsymbol{\kappa}){\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}))\omega^2 n}{8}\big(\mathcal{W}_{q,n+1}+\mathcal{W}_{q,n-1}-2\mathcal{W}_{q,n}\big) \\ & -c_0^2\kappa_q(\boldsymbol{\kappa})\sigma_q(\boldsymbol{\kappa})\mathcal{W}_{q,n},\end{aligned} \tag{6.9}$$

with $\mathcal{W}_{q,0}(\omega,\boldsymbol{\kappa},s) = \delta(s)$. For $n\geq 1$, they are given by

$$\mathcal{W}_{q,n}(\omega,\boldsymbol{\kappa},s) = e^{-c_0^2\kappa_q(\boldsymbol{\kappa})\sigma_q(\boldsymbol{\kappa})s} c_0^2\kappa_q {\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}))\omega^2 P_n^\infty\big(c_0^2\kappa_q(\boldsymbol{\kappa}){\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}))\omega^2 s\big), \tag{6.10}$$

with

$$P_n^\infty(x) = \frac{8nx^{n-1}}{(8+x)^{n+1}}\mathbf{1}_{[0,\infty)}(x). \tag{6.11}$$

This gives for $\tau>0$:

$$\begin{aligned}&\frac{1}{\delta}\mathbb{E}\Big[H_{{\rm ref},q}\big(\frac{\tau}{\delta}\big)H_{{\rm ref},q}\big(\frac{\tau}{\delta}+t\big)\Big] \xrightarrow{\delta\to 0} \frac{(1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)^2}{2\pi} \\ &\times \int \frac{8e^{-\frac{\sigma_q(\boldsymbol{\kappa}_{\rm inc})}{\kappa_q(\boldsymbol{\kappa}_{\rm inc})}\tau}}{\big(8+(1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)\frac{{\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}_{\rm inc}))\omega^2}{\kappa_q(\boldsymbol{\kappa}_{\rm inc})}\tau\big)^2}\frac{{\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}_{\rm inc}))\omega^2}{\kappa_q(\boldsymbol{\kappa}_{\rm inc})}|\hat{f}_q(\omega)|^2 e^{i\omega t}d\omega.\end{aligned} \tag{6.12}$$

This expression means that:
1) Around time τ/δ, the field is locally stationary in time and its local power spectral density (or Wigner transform) defined by

$$S_q(\omega;\tau) = \lim_{\delta\to 0}\frac{1}{\delta}\int \mathbb{E}\Big[H_{{\rm ref},q}\big(\frac{\tau}{\delta}\big)H_{{\rm ref},q}\big(\frac{\tau}{\delta}+t\big)\Big]e^{-i\omega t}dt$$

is

$$S_q(\omega;\tau) = \frac{8(1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)^2 e^{-\frac{\sigma_q(\boldsymbol{\kappa}_{\rm inc})}{\kappa_q(\boldsymbol{\kappa}_{\rm inc})}\tau}}{\big(8+(1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)\frac{{\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}_{\rm inc}))\omega^2}{\kappa_q(\boldsymbol{\kappa}_{\rm inc})}\tau\big)^2}\frac{{\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}_{\rm inc}))\omega^2}{\kappa_q(\boldsymbol{\kappa}_{\rm inc})}|\hat{f}_q(\omega)|^2. \tag{6.13}$$

For any central time τ the power spectral density of the reflected wave presents a maximum at some frequency that is a decaying function of τ. If the spectrum of the incoming wave is very large, if there is no dissipation, and if $n\equiv 0$, then $\gamma_q(\omega,\boldsymbol{\kappa}_{\rm inc})$ does not depend on ω, it is given by (5.34-5.35), and the power spectral density is maximal at the frequency $\omega_{{\rm max},q}$ given by

$$\omega^2_{{\rm max},q}(\boldsymbol{\kappa}_{\rm inc},\tau) = \frac{8\kappa_q(\boldsymbol{\kappa}_{\rm inc})}{(1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)\gamma_q(\boldsymbol{\kappa}_{\rm inc})\tau}. \tag{6.14}$$

2) The mean reflected intensity at time τ/δ is

$$\frac{1}{\delta}\mathbb{E}\Big[H_{{\rm ref},q}\big(\frac{\tau}{\delta}\big)^2\Big] \stackrel{\delta\to 0}{\longrightarrow} \frac{(1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)^2}{2\pi} \times \int \frac{8 e^{-\frac{\sigma_q(\boldsymbol{\kappa}_{\rm inc})}{\kappa_q(\boldsymbol{\kappa}_{\rm inc})}\tau}}{\Big(8+(1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)\frac{{\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}_{\rm inc}))\omega^2}{\kappa_q(\boldsymbol{\kappa}_{\rm inc})}\tau\Big)^2} \frac{{\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}_{\rm inc}))\omega^2}{\kappa_q(\boldsymbol{\kappa}_{\rm inc})} |\hat{f}_q(\omega)|^2 d\omega. \tag{6.15}$$

The mean reflected intensity slowly decays in time as τ^{-2} if there is no dissipation. This long power delay spread is characteristic of the localization regime encountered in one-dimensional or in three-dimensional randomly layered media [18].

The total reflectivity at frequency ω is defined by

$$\mathcal{R}^2_{{\rm tot},q}(\omega) = \lim_{\delta\to 0} \frac{\mathbb{E}[|\hat{H}_{{\rm ref},q}(\omega)|^2]}{|\hat{f}_q(\omega)|^2}, \tag{6.16}$$

which can computed by integrating (6.12):

$$\mathcal{R}^2_{{\rm tot},q}(\omega) = \lim_{\delta\to 0} \frac{1}{|\hat{f}_q(\omega)|^2} \iint \frac{1}{\delta}\mathbb{E}\Big[H_{{\rm ref},q}\big(\frac{\tau}{\delta}\big)H_{{\rm ref},q}\big(\frac{\tau}{\delta}+t\big)\Big] e^{-i\omega t} d\tau dt.$$

It is given by

$$\mathcal{R}^2_{{\rm tot},q}(\omega) = R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2 + (1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)\psi\Big(\frac{8\sigma_q(\boldsymbol{\kappa}_{\rm inc})}{(1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2){\rm Re}(\gamma_q(\omega,\boldsymbol{\kappa}_{\rm inc}))\omega^2}\Big), \tag{6.17}$$

$$\psi(x) = 1 - x e^x {\rm E}_1(x), \tag{6.18}$$

and ${\rm E}_1$ is the exponential integral function ${\rm E}_1(x) = \int_x^\infty \frac{e^{-u}}{u} du$. We have $\psi(x) = 1/x + O(1/x^2)$ as $x\to\infty$ and $\psi(x) = 1 + x\ln x + o(x\ln x)$ as $x\to 0^+$. Note that we can also write in terms of the dissipation length (5.31) and the localization length (5.32):

$$\mathcal{R}^2_{{\rm tot},q}(\omega) = R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2 + (1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)\psi\Big(\frac{2L_{{\rm loc},q}(\omega,\boldsymbol{\kappa}_{\rm inc})}{(1-R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2)L_{{\rm dis},q}(\omega,\boldsymbol{\kappa}_{\rm inc})}\Big). \tag{6.19}$$

If there is no dissipation and the bulk medium is scattering, then $\mathcal{R}^2_{{\rm tot},q}(\omega) = 1$ for all ω, which means that the whole wave energy is reflected, as predicted by localization theory.
If there is no random scattering in the bulk medium, then $\mathcal{R}^2_{{\rm tot},q}(\omega) = R^{\rm i}_q(\boldsymbol{\kappa}_{\rm inc})^2$ for all ω, which simply means that only the interface generates reflections.
An example showing the total reflectivity for the p,s-waves is given in Figure 6.1 in the case without small and slow fluctuations. For the total reflectivity to be high, we need the interface reflectivity to be high or the localization length to be smaller than the dissipation length. In the latter case the wave is reflected by the bulk medium before being attenuated (see Figure 6.3). If the localization length is larger than the dissipation length, then the wave spends too much time in the bulk medium before being reflected and it is then dissipated (see Figure 6.2).

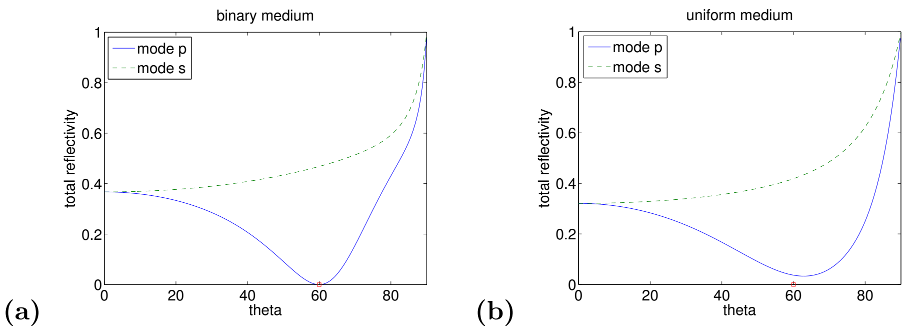


FIG. 6.1. *Total reflectivities* $\mathcal{R}^2_{\mathrm{tot},q}(\omega)$ *for the two models and the* p,s*-waves. Here* $\mu_0=1$, $\varepsilon_0=1$, $\varepsilon_1=3$, $\bar{\varepsilon}_{\mathrm{d}}=0.01$, $\alpha=0.1$, $l_{\mathrm{c}}=1$, $\omega=1$.

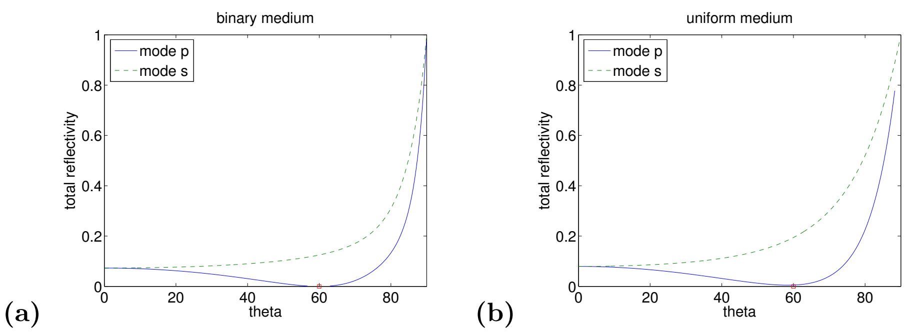


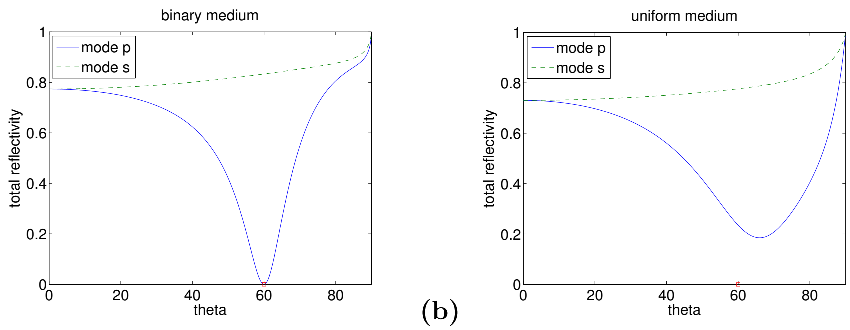
FIG. 6.2. *Same as in Fig. 6.1 but with* $\bar{\varepsilon}_{\mathrm{d}}=0.1$.

7. Conclusion We have analyzed some phenomena related to electromagnetic wave reflection by a randomly layered half-space. Both homogenization and diffusion approximation results are needed to capture the interface reflectivity and the bulk medium reflectivity. The main result is that there are critical incidence angles for which the interface reflectivity or the bulk medium reflectivity can become very small or even vanish. However these two critical angles are in general distinct so that it is not possible to minimize simultaneously the interface and bulk medium reflectivities.

The results described in this paper can be readily extended to other frameworks, even for scalar waves, for instance for acoustic waves [6, Section 17.3]:

$$\rho(z)\partial_t \boldsymbol{u} + \nabla p = 0,$$
$$\frac{1}{K_0}\partial_t p + \nabla\cdot\boldsymbol{u} = 0,$$

where $\boldsymbol{u}(t,\boldsymbol{x},z)$ is the three-dimensional velocity field, $p(t,\boldsymbol{x},z)$ is the pressure field, K_0 is the bulk modulus of the medium (assumed to be constant), and $\rho(z)$ is the density of the medium (assumed to be fluctuating only in the z-variable). The existence of a Brewster angle and of a generalized Brewster angle can be obtained in the same way as described in this paper. For instance, in the case of of a binary medium made of materials with densities ρ_0 and ρ_1, the Brewster and generalized Brewster angles are



(a) (b)

FIG. 6.3. *Same as in Fig. 6.1 but with* $\bar{\varepsilon}_{\rm d} = 0.001$.

equal and given by

$$|\boldsymbol{\kappa}_{\rm B}|^2 c_0^2 = \frac{\rho_1}{\rho_0 + \rho_1},$$

with $c_0^2 = K_0/\rho_0$. This means that a plane wave incoming from a homogeneous half-space with parameters (K_0, ρ_0) onto a randomly layered medium made of materials with parameters (K_0, ρ_0) and (K_0, ρ_1) is fully transmitted if its incidence angle is $\theta_{\rm B} = \arctan(\sqrt{\rho_1/\rho_0})$. Furthermore, as in the electromagnetic case addressed in this paper, the binary medium model is the only one for which it is possible to cancel both the interface reflectivity and the bulk medium reflectivity for some incidence angle.

Finally, we remark that, in electromagnetics, it is possible to exhibit Brewster and generalized Brewster anomalies when the permittivity ε is constant while the permeability μ is spatially varying. However, in acoustics, there is neither Brewster nor generalized Brewster anomaly when the density ρ is constant while the bulk modulus K is spatially varying [6, Section 17.1].

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